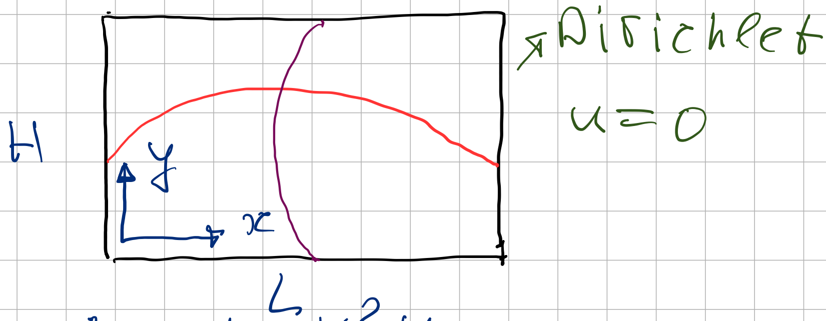
Lecture 10. Resonators.
Rayleigh quotient

1) Rectangular resonator



$$\Delta u + k_{m,n}^2 u = 0$$

Via separations of variables

$$u = \sum_{m,n=1}^{\infty} A_{m,n} \sin \frac{\pi m}{L} x \sin \frac{\pi n}{H} y$$

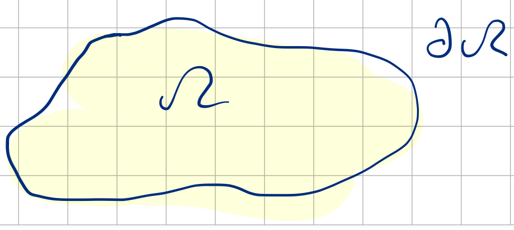
Resonator modes only exist only for discrete set of $k_{m,n}$ such that

$$\left(\frac{\pi m}{L}\right)^2 + \left(\frac{\pi n}{H}\right)^2 = k_{m,n}^2 ;$$

$k_{m,n}^2$ are eigenvalues of the problem. Modes are orthogonal, any field can be decomposed into a sum of modes.

2) General case

Discrete eigenvalue problem



$$\Delta u + k^2 u = 0$$

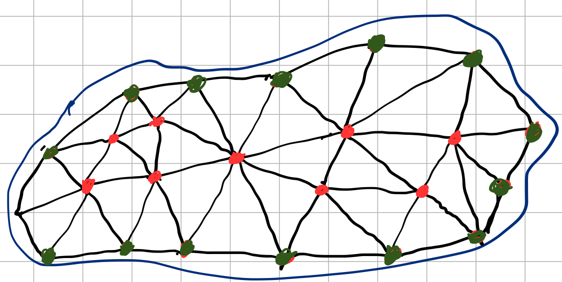
$u = 0$ on $\partial\Omega$

Find κ_j and u_j that satisfy the problem.

General approach is to define an associated operator (functional) and study its eigenvalues and eigen vectors (stationary points and minimizing functions).

We study simpler discrete analogue.

Let's divide our domain into finite elements. Define a vector of nodal field values U of size N.



Using finite element method (FEM) the problem is reduced to generalised eigenvalue problem:

$$\boxed{K U_j = \kappa_j^2 M U_j}$$

K – stiffness matrix

M – mass matrix, positively defined

Both are real symmetric, of size $N \times N$

Let's build some estimates for k_j using functional analysis.

3) Rayleigh inequality (quotient)

Let A be hermitian:

$$A = A^{+} \equiv (A^{*})^{T}$$

A of size $N \times N$

Consider eigenvalue problem

$$Ax = \lambda x$$

$x \in \mathbb{C}^{N}$, vector

Properties

a) λ_j are real:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

b) N different eigenvectors x_j exist. Eigenvectors are ortogonal:

$$x_j^{+} x_m = 0, \quad j \neq m$$

λ_j and λ_m may coincide

Thus x_j form ortogonal basis in $\mathbb{C}^{N}$

Proposition (Rayleigh quotient).

$$\lambda_1 = \min_{x} \frac{x^{+} A x}{x^{+} x},$$

smallest eigenvalue

Note that it's invariant to $x \to bx$

In more complicated cases $\min_x$ is replaced with $\inf_x$

Proof. Expand arbitrary x:

$$x = \sum_{n=1}^{N} c_n x_n$$

$$\frac{x^+ A x}{x^+ A x} = \frac{\sum_{n=1}^{N} \lambda_n |c_n|^2 x_n^+ x_n}{\sum_{n=1}^{N} |c_n|^2 x_n^+ x_n}$$

Obviously,

$$\sum_{n=1}^{N} \lambda_n |c_n|^2 x_n^+ x_n \geqslant \lambda_1 \sum_{n=1}^{N} |c_n|^2 x_n^+ x_n$$

Thus

$$\lambda_1 \leq \min_x \frac{x^+ A x}{x^+ x}$$

For $x = x_1$ we have equality. QED

Applications:

a) Find λ_1 through optimization

b) Construct test function x to show that λ_1 less than some threshold

Theorem (Courant-Fisher minimax theorem)

$$\lambda_p = \min_{L_p} \max_{x \in L_p} \frac{x^+ A x}{x^+ x}$$

arbitrary subspace of $\mathbb{C}^N$ of size p

Comments.

$$P(L_p) = \max_{x \in L_p} \frac{x^+ A x}{x^+ x}$$

computed for all possible L_p

Then a minimum of $P(L_p)$ across all L_p is found.

For $p=1$ we get Rayleigh quotient

Can be prooved in a similar fashion.

4) Generalised eigenvalue problem

$$A x = \lambda B x$$

hermitian

positevely defined

$x^+ B x > 0, \; x \neq 0$

Note that

$$B^{-1} A x = \lambda x$$

not hermitian

Instead use Cholesky decomposition:

$$B = L L^+$$

Lower triangular with positive values on diagonale

Upper triangular

$$L = \begin{pmatrix} l_{11} & & & 0 \\ \bullet & l_{22} & & \\ \bullet & \bullet & \ddots & \\ \bullet & \bullet & \cdots & l_{nn} \end{pmatrix}$$

positive real

Then

$$Ax = \lambda L L^{+} x$$

$$\tilde{A} y = \lambda y$$

$$y = L^{+} x, \qquad \tilde{A} = L^{-1} A (L^{+})^{-1}$$

Hermitian

Rayleigh quotient for generalised eigenvalue problem:

$$\lambda_1 = \min_{x} \frac{x^{+} A x}{x^{+} B x}$$

$$\lambda_p = \min_{L_p} \max_{x \in L_p} \frac{x^{+} A x}{x^{+} B x}$$

5) Application to FEM problem

$$k_1^2 = \min_{U} \frac{U^{+} K U}{U^{+} M U}$$

Remark. For Neumann boundary condition $k_1 = 0$ for $U = \text{const.}$ But

$$k_p^2 = \min_{L_p} \max_{x \in L_p} \frac{U^{+} K U}{U^{+} M U}$$

still makes sense.

6) Continuous problem

Let's guess Rayleigh quotient for

$$\Delta u = k_j^2 u$$

$$u \to U$$

$$U^{+} M U \to \iint_{\Omega} |u|^2 ds$$

$$U^{+} K U \to \iint_{\Omega} |\nabla u|^2 ds$$

Thus

$$k_1^2 = \min_{u(x)} \frac{\iint_{\Omega} |\nabla u|^2 ds}{\iint_{\Omega} |u|^2 ds},$$

where $u(x)$ is a test function

$u(x)$ belongs to space of functions that have derivatives almost every where (aka Sobolev space H^1). For Dirichlet boundary condition $u(x)=0$ for $x \in \partial\Omega$

$$k_p^2 = \min_{L_p} \max_{u(x) \in L_p} \frac{\iint_{\Omega} |\nabla u|^2 ds}{\iint_{\Omega} |u| ds}$$

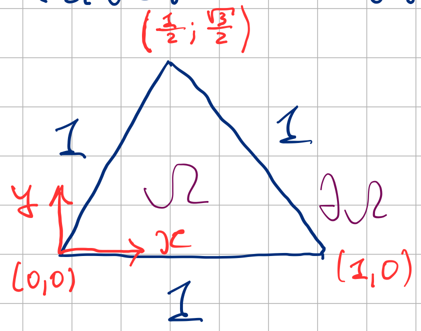
Can be prooved similarly to discrete version:

a) Introduce basis $u_j(x)$ that corresponds to $\lambda_j = k_j^2$

b) Use ortogonality:

$$\iint_{\Omega} u_j(x) u_m(x) ds = 0, \quad j \neq m$$

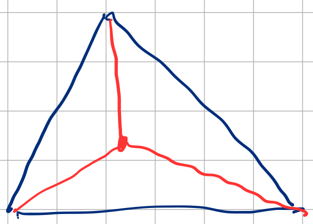
7) Example. First eigenvalue of equilateral triangle



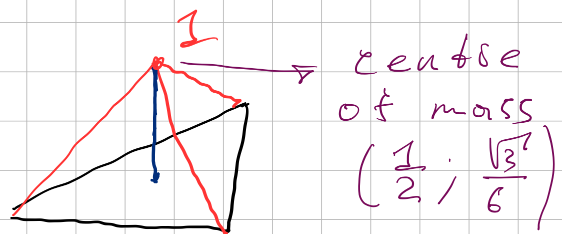
$$\Delta u + k^2 u = 0$$

$$u\big|_{\partial\Omega} = 0$$

Test function: piecewise linear



Top view



side view

$$u = 3\min\left\{1 - x - \frac{y}{\sqrt{3}}, x - \frac{y}{\sqrt{3}}, \frac{2y}{\sqrt{3}}\right\}$$

$$\iint_\Omega u^2\, ds = \frac{\sqrt{3}}{24}$$

$$\iint_\Omega |\nabla u^2|\, ds = 3\sqrt{3}$$

$$K_1^2 \leq \frac{\iint_\Omega |\nabla u|^2\, ds}{\iint_\Omega u^2\, ds} = 72$$

Rigorous analysis gives

$$K_1^2 = \frac{4\pi}{\sqrt{3}} \approx 52.63$$

First mode is a combination of 6 plane waves