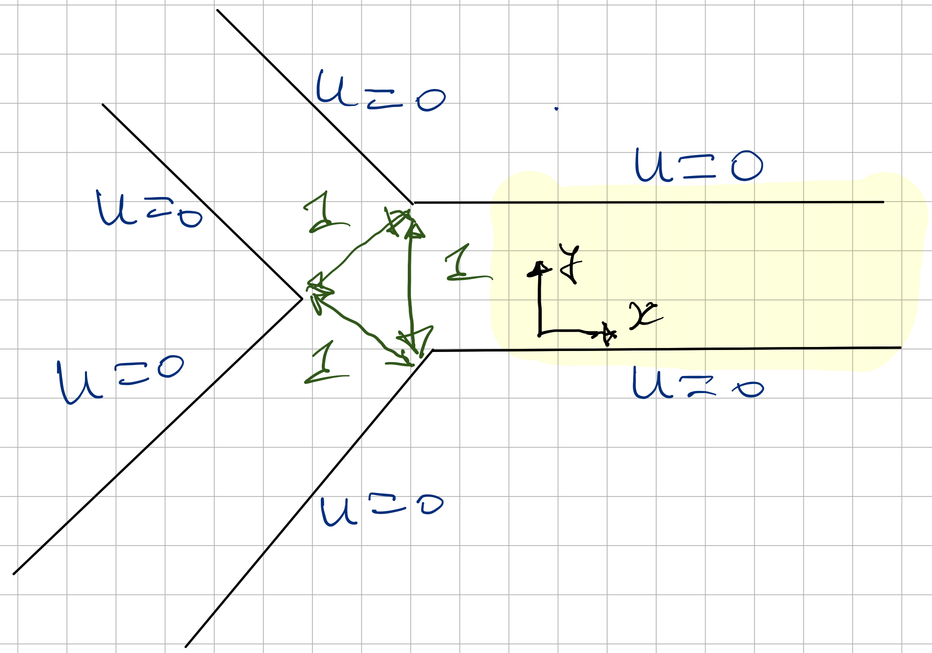
Lecture 11. Rayleigh inequality.

Example 2. Localised modes in a joining of 3 waveguides.



Waveguide modes can be propagating or exponentially decaying:

$$u_n = \sin(\pi n y)\, e^{i\sqrt{k^2-\pi^2 n^2}\,x};$$

$n = 1, 2, \ldots$

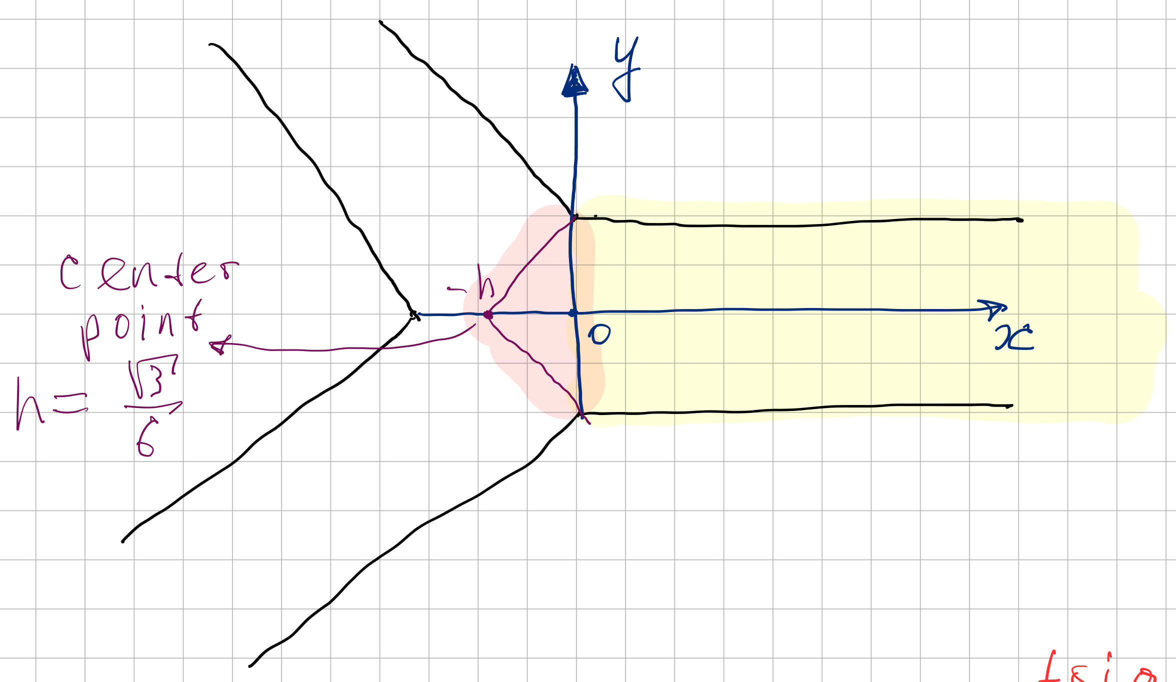
if $k < \pi$, there are no propagating modes.

Let's show that a localised (trapped) mode can exist in such a system. (This result was presented by S. A. Nazarov 27.02.2024 in Saint-Petersburg's branch of Steklov's institute)

To prove the existence we will build a test function $\tilde{u}(x)$ for which Rayleigh quotient is less than π^2.

Remark. Test function also should decay exponentially in waveguides.

a) Let $\tilde{u}(x)$ be symmetric with respect to rotation $\frac{2\pi}{3}$



Let

triangular part

$$\tilde{u}(x,y) = \begin{cases} \cos(\pi y) & -h < x < 0, \ |y| \le (x+h)\sqrt{3} \\ \cos(\pi y) e^{-dx}, & x \ge 0, \ |y| < \frac{1}{2} \end{cases}$$

waveguide part

Denote triangular and waveguide part as Δ and $\sqsubset$

d is to be defined

b) Compute Rayleigh quotient:

$$R = \frac{\iint_\Omega |\nabla u|^2 ds}{\iint_\Omega |u| ds} = \frac{a_\Delta + a_\sqsubset}{b_\Delta + b_\sqsubset}$$

$$a_\Delta = \iint_\Delta |\nabla u|^2 ds = \frac{\pi^2 - 4}{8\sqrt{3}}$$

$$b_\Delta = \iint_\Delta |u|^2 ds = \frac{\pi^2 + 4}{8\sqrt{3}\,\pi^2}$$

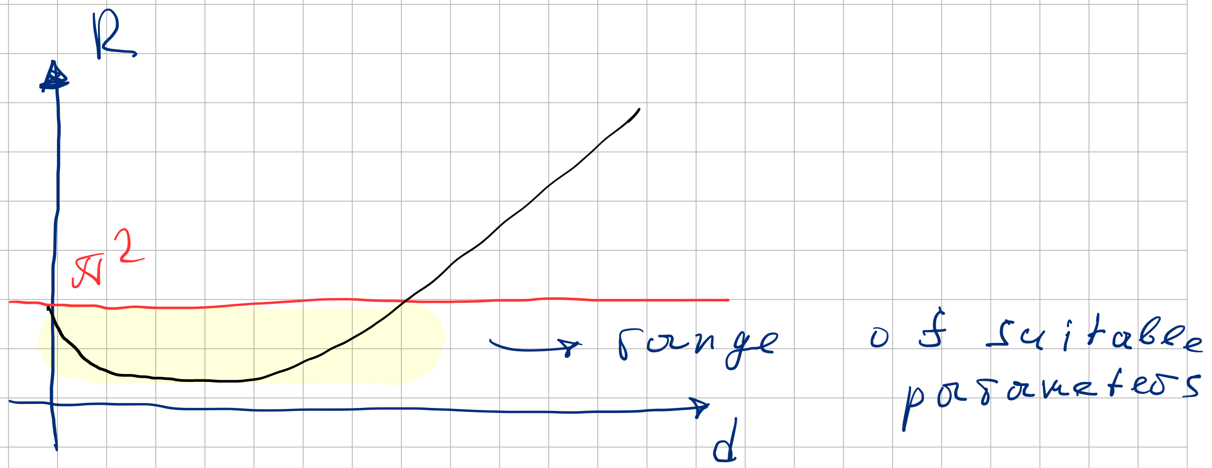
$$\boxed{\frac{a_\Delta}{b_\Delta} < \pi^2}$$

$$a_\sqsubset = \frac{\pi^2 + d^2}{4d}; \quad b_\sqsubset = \frac{1}{4d};$$

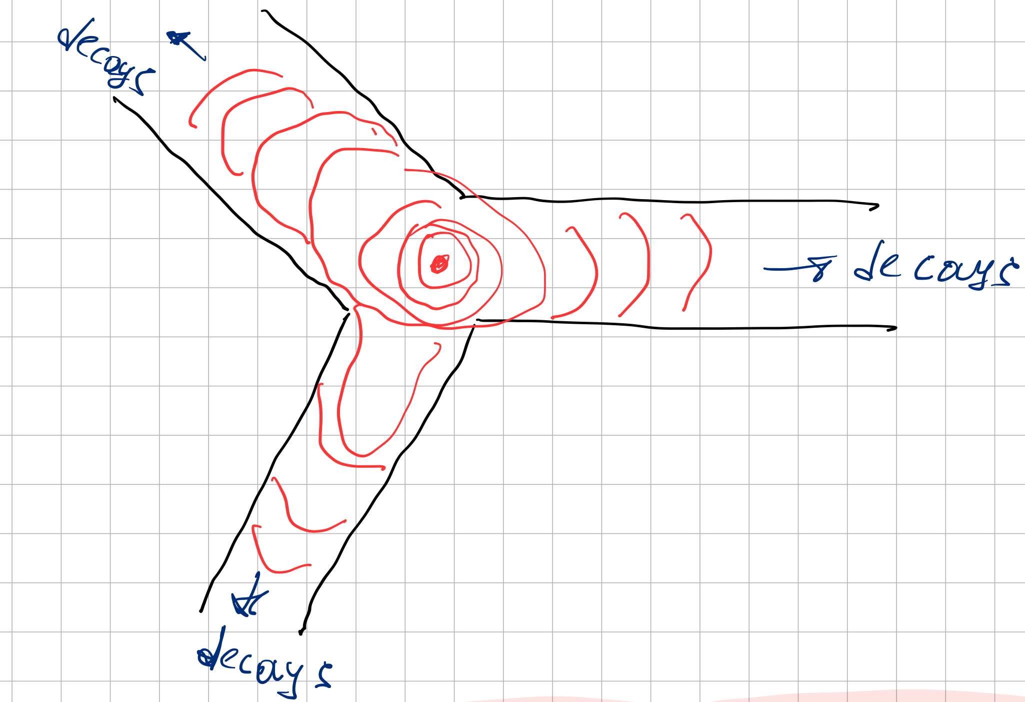
Let $d = 1$

$$R = \frac{\frac{\pi^2 - 4}{8\sqrt{3}} + \frac{\pi^2 + 1}{4}}{\frac{\pi^2 + 4}{8\sqrt{3}\,\pi^2} + \frac{1}{4}} = \frac{\pi^2 - 4 + 2\sqrt{3}(\pi^2 + 1)}{1 + \frac{4}{\pi^2} + 2\sqrt{3}} \simeq 9 < \pi^2$$

QED



Mode should look like this:



Homogenisation

Consider an equation:

$$\operatorname{div}\left(q\left(\frac{\vec{x}}{\varepsilon}\right)\nabla u(\vec{x})\right) + \Lambda^2 u(\vec{x}) = 0$$

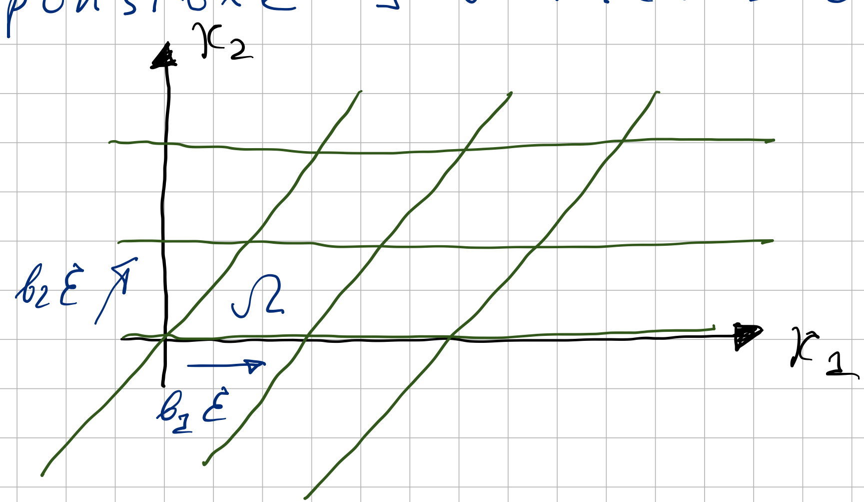
$\vec{x} = (x_1, x_2)$, ε – small parameter

Λ – constant

q is periodic:

$$q(\vec{y} + \vec{b}_1) = q(\vec{y} + \vec{b}_2) = q(y)$$

responsible for microscopic scale



Example. If u is anti-plane deformation of a solid, then

g — shear modulus

$\Lambda^2 = \rho\omega^2$; ρ — density

The problem:

Build a homogenized equation for $u(x)$ that descibes macroscopic behaviour of the field.

Background. Method of multiple scales.

The problem has two scales:

a) macroscopic $\vec{x} \sim 1$

b) microscopic $\vec{x} \sim \varepsilon$.

Introduce two formal variables:

$$X = x = (x_1, x_2)$$

$$\xi = \frac{x}{\varepsilon}$$

Look for the solution in the form:

$$u(X, \xi) = u^0(X, \xi) + \varepsilon u^1(X, \xi)$$

In the end

$$u(x) = u\left(x, \frac{x}{\varepsilon}\right);$$

Additional "periodicity" restriction

$$u^m(X, \xi + b_1) = u^m(X, \xi + b_2) = u^m(X, \xi)$$

Implementation

For arbitrary $f(X, \xi)$

$$\nabla f = \nabla_X f + \varepsilon^{-1}\nabla_\xi f$$

$$\nabla u(X,\xi) = \varepsilon^{-1}\nabla_\xi u^0 + \varepsilon^0(\nabla_X u^0 + \nabla_\xi u^1) + \bar{O}(\varepsilon)$$

Note that $q = q(\xi)$

Substitue the latter into governing equation:

ε^{-2}:

$$\operatorname{div}_\xi(q\nabla_\xi u^0) = 0$$

ε^{-1}:

$$\operatorname{div}_X(q\nabla_\xi u^0) + \operatorname{div}_\xi(q\nabla_X u^0) + \operatorname{div}_\xi(q\nabla_\xi u^1) = 0$$

ε^0:

$$\operatorname{div}_\xi(q\nabla_\xi u^2) + \operatorname{div}_\xi(q\nabla_X u^1) + \operatorname{div}_X(q\nabla_\xi u^1) + \operatorname{div}_X(q\nabla_X u^0) + \Lambda^2 u^0 = 0$$

Let's study the equations order by order.

a) Order ε^{-2}

$$u^0(X,\xi) = u^0(X)$$

is a solution

It can be proved usig Gauss theorem that there are no other solutions

Let's build equation for u^0 using equation for orders ε^{-1} and ε^0

b) Order ε^{-1}

"Known function"

$$\operatorname{div}_\xi (q \nabla_\xi u^1) = -\sum_{j=1}^{2} \frac{\partial u^0}{\partial x_j} \frac{\partial q}{\partial \xi_j}$$

Consider it as equation of ξ

State an auxiliary problem:

Find $N_j(\xi)$ such that

$$\operatorname{div}_\xi (q(\xi) \nabla_\xi N_j(\xi)) = -\frac{\partial q(\xi)}{\partial \xi_j}$$

Then

$$u^{(1)}(x, \xi) = \sum_{j=1}^{2} \frac{\partial u^0}{\partial x_j} N_j(\xi)$$

c) Order ε^0

Let's average the equation over Ω

Denote averaging with brackets:

$$\langle f(x, \xi) \rangle = \frac{1}{S} \iint_{\Omega} f(x, \xi)\, ds$$

S – area of the domain

div_ξ terms do not contribute

$$\operatorname{div}_x (\langle q \rangle \nabla_x u^0) + \operatorname{div}_x \langle q \nabla_\xi u^1 \rangle + \lambda^2 u^0 = 0$$

$$\operatorname{div}_x \langle q \nabla_\xi u^{(1)} \rangle = \sum_{j=1}^{2} \sum_{m=1}^{2} \frac{\partial}{\partial x_j} \left\langle q(\xi) \frac{\partial N_m(\xi)}{\partial \xi_j} \right\rangle \frac{\partial u^0(x)}{\partial x_m}$$

Denote

$$Q_{j,m} = \langle q(\xi) \rangle \delta_{j,m} + \left\langle q(\xi) \frac{\partial N_m(\xi)}{\partial \xi_j} \right\rangle$$

Thus, order ε^0 gives

$$\sum_{j=1}^{2} \sum_{m=1}^{2} \frac{\partial}{\partial X_j} Q_{j,m} \frac{\partial}{\partial X_m} u^0(X) + \Lambda^2 u^0(X) = 0$$

$$(*) \quad \boxed{\mathrm{div}_X \left(Q \nabla_X u^0(X) \right) + \Lambda^2 u^0(X) = 0}$$

Q - tensor that contains all microscopic behaviour

$$(**) \quad \mathrm{div}_\xi \left(q(\xi) \nabla_\xi N_j(\xi) \right) = - \frac{\partial q(\xi)}{\partial \xi_j}$$

This equation is solved on one cell