Diffraction theory
Lecture 1

Governing equations

a) Linearized equations of Hydrodynamics

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\nabla p \qquad (*)$$

$$\frac{\partial p}{\partial t} = -\rho c^2 \operatorname{div} \vec{v} + \rho c^2 f \qquad (**)$$

Introduce acoustic potential

$$p = -\rho \frac{\partial u}{\partial t}; \quad \vec{v} = \nabla u$$

$(*)$ satisfied automatically

$(**)$ reduces to

$$\boxed{\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f.}$$

Consider harmonic problem

$$u \sim e^{-i\omega t} \qquad f \sim e^{-i\omega t}$$

$$\Delta u + k^2 u = f$$

$$k = \frac{\omega}{c}$$

Note that $e^{ikr - i\omega t}$ "outgoing"

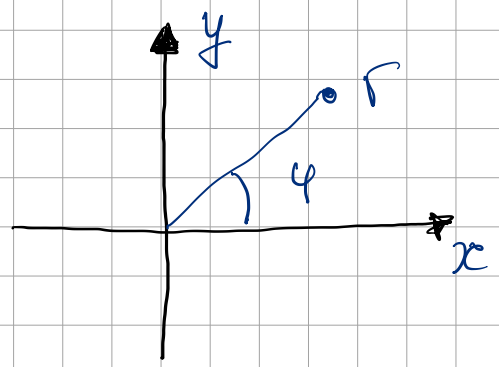
$e^{-ikr - i\omega t}$ "incoming"

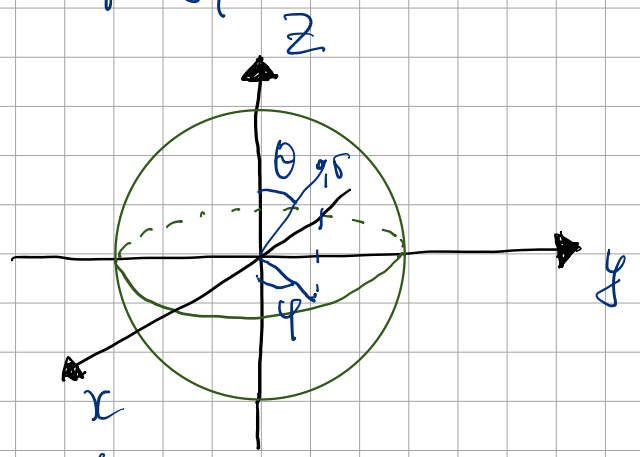
$$p = i\rho\omega u$$

b) Let's recall structure of Δ

In 2D polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$





$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\theta,\varphi}$$

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

c) Green's function of Helmholtz equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}} + kU = \delta(x)\delta(y)$$

$\downarrow$

Bessel equation

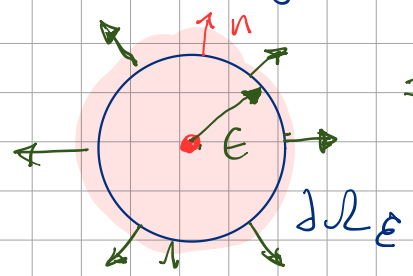
$$U = A H_0^{(1)}(kr) + \cancel{B H_0^{(2)}(kr)}$$

$$H_0^{(1)}(z) \sim \frac{e^{iz}}{\sqrt{z}} \qquad H_0^{(2)}(z) \sim \frac{e^{-iz}}{\sqrt{z}}$$

Thus $B=0$; Let's find A

$$H_0^{(1)}(z) = \frac{2i}{\pi}\ln\frac{z}{2} + O(1)$$

Let's integrate the equation



flow from the source

$$\int_{\Omega_\varepsilon}\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right)dS=\int_0^{\varepsilon}\int_0^{2\pi}\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)r\,dr\,d\varphi=$$

$$=2\pi\varepsilon\left.\frac{du}{dr}\right|_{r=\varepsilon}$$

$$2\pi\varepsilon\left.\frac{du}{dr}\right|_{r=\varepsilon}+k^2\int_{\Omega_\varepsilon}u\,dS=1$$

$$A\,2\pi\varepsilon\frac{2i}{\pi}\frac{1}{\varepsilon}+O(1)=1$$

$$u(r)\equiv G(r)=-\frac{i}{4}H_0^{(1)}(kr)$$

Similarly, in 3D:

$$G(r)=-\frac{e^{ikr}}{4\pi r}$$

d) Plane waves

Let k and θ be real

$$w(x,y,\theta)=e^{ik(x\cos\theta+y\sin\theta)}$$

$$w(r,\varphi,\theta)=e^{ikr\cos(\varphi-\theta)}$$

Note that we can make k and θ complex

Denote $\xi=k\cos\theta$

$$w(x,y,\xi)=e^{i\xi x+\sqrt{k^2-\xi^2}\,y}$$

ξ - variable of Fourier transform

if K and ξ real, then

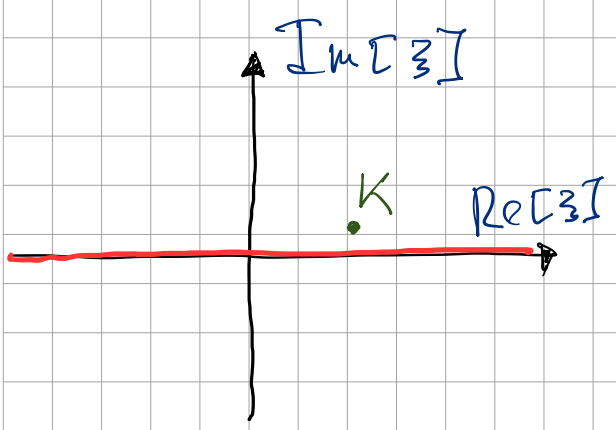
if $|\xi| \leq K$ we have a usual plane wave. If $|\xi| > K$, plane wave decayes along y-axis.

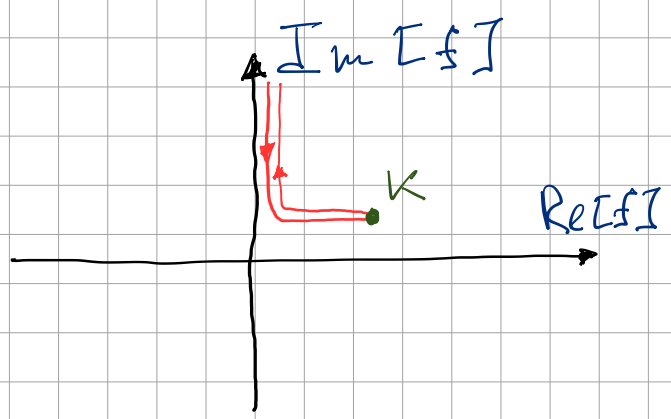
Choose the branch $\sqrt{}$ such that it is close to positive real/imaginary values.

To formally distinguish between outgoing and incoming waves we will consider

$$K = K' + iK'', \quad K'' - \text{positive small}$$

Let's find an image of real line ξ: $f(\xi) = \sqrt{K^2 - \xi^2}$





$$\mathrm{Im}[f] > 0$$

e) Derivation of Green's function via Fourier transform

$$\Delta u + k^2 u = \delta(x)\,\delta(y)$$

$$k = k' + i k'', \quad k'' \to 0$$

Forward transform:

$$u(x,y) \xrightarrow{\mathcal{F}} \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} u(x,y)\, e^{-i\xi_1 x - i\xi_2 y}\, dx\,dy$$

$$= \tilde{u}(\xi_1, \xi_2)$$

$$\tilde{u}(\xi_1, \xi_2) \xrightarrow{\mathcal{F}^{-1}} \iint_{-\infty}^{\infty} \tilde{u}(\xi_1, \xi_2)\, e^{i\xi_1 x + i\xi_2 y}\, d\xi_1\, d\xi_2$$

Diff rule:

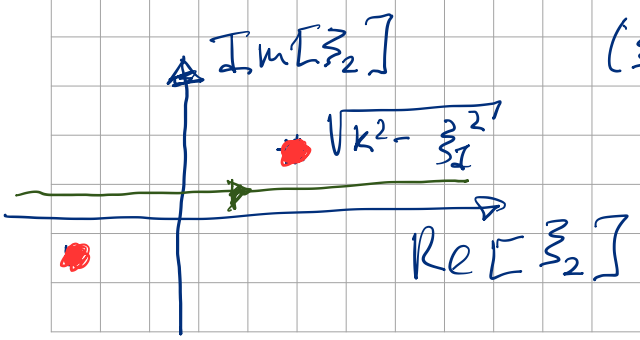
$$\frac{\partial u}{\partial x} \to i\xi_1 \tilde{u}(\xi_1, \xi_2)$$

$$\left[-(\xi_1^2 + \xi_2^2) + k^2\right] u(\xi_1, \xi_2) = \frac{1}{2\pi^2}$$

$$u(\xi_1, \xi_2) = -\frac{1}{(2\pi)^2} \frac{1}{\xi_1^2 + \xi_2^2 - k^2}$$

$$\boxed{u(x,y) = -\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{e^{i\xi_1 x + i\xi_2 y}}{\xi_1^2 + \xi_2^2 - k^2}\, d\xi_1\, d\xi_2}$$

$$\frac{1}{\left(\xi_2 + \sqrt{k^2 - \xi_1^2}\right)\left(\xi_2 - \sqrt{k^2 - \xi_1^2}\right)}$$

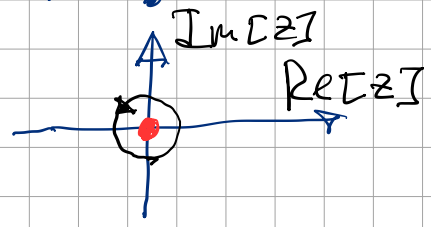


Let $y > 0$

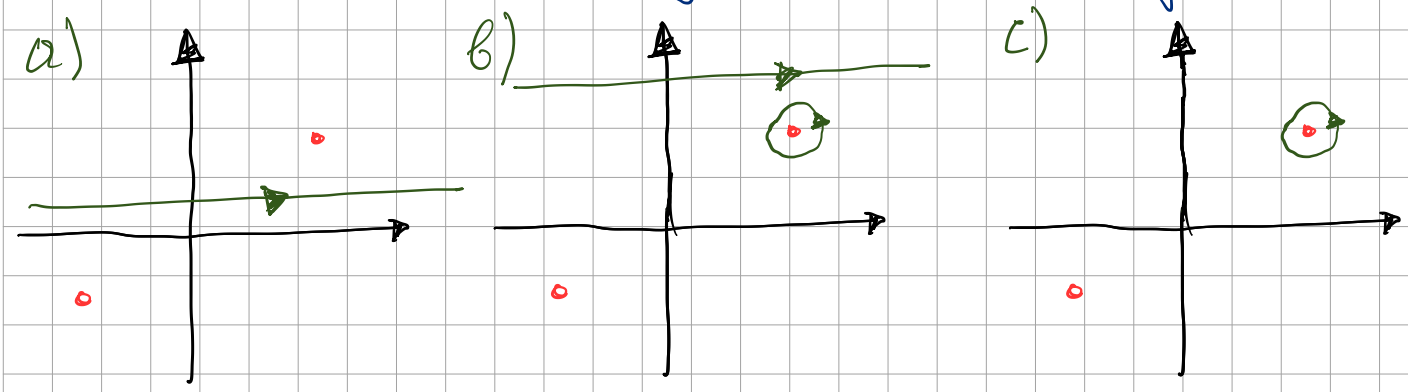
Remark. Basics of complex analysis

1) Residue integration:

$$\int_\gamma \frac{f(z)\,dz}{z} = 2\pi i f(0)$$



2) In the domain of analyticity of the integrand the contour of integration can be deformed freely – Cauchy theorem



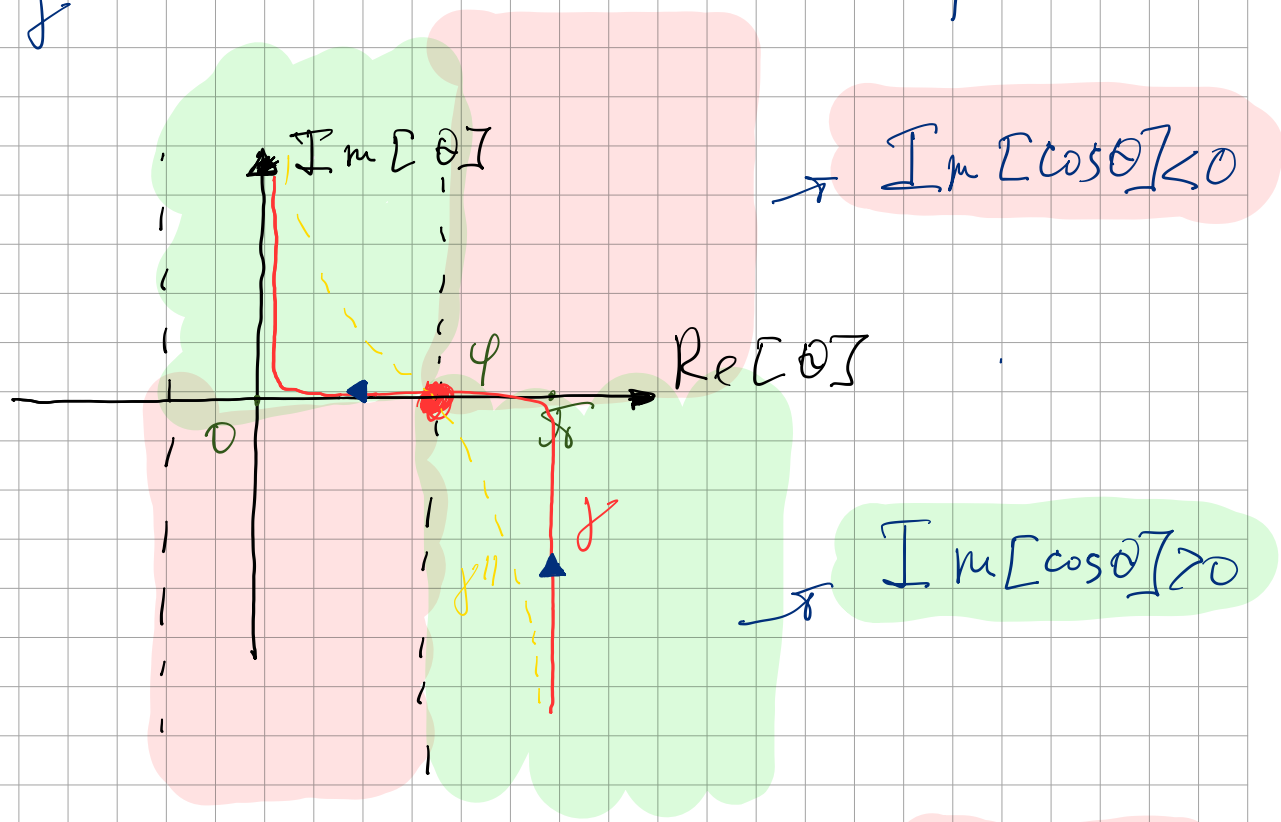
Between b) and c) we used Jordan Lemma

$$u(x,y) = \frac{-2\pi i}{(2\pi)^2}\int_{-\infty}^{\infty} \frac{e^{i\xi_1 x + i\sqrt{k^2-\xi_1^2}\,y}}{2\sqrt{k^2-\xi_1^2}}\,d\xi_1$$

$$\theta: \begin{cases} \xi = k\cos\theta \\ \sqrt{k^2-\xi^2} = \sin\theta \end{cases}$$

$$u(r,\varphi) = \frac{1}{4\pi i}\int_\gamma \frac{e^{ikr\cos(\theta-\varphi)}}{k\sin\theta}\,dk\cos\theta =$$

$$= \frac{i}{4\pi} \int_{\gamma} e^{ik r \cos(\theta - \varphi)} \, d\theta = -\frac{i}{4} H_0^{(1)}(kr)$$



Asymptotic estimation $kr \gg 1$

Let $\tau = (\theta - \varphi) e^{\frac{i\pi}{4}}$ $\quad \left(e^{\frac{i\pi}{4}}\right)^2 = i$

$$u = -\frac{i e^{-\frac{i\pi}{4}}}{4\pi} \int_{-\infty}^{\infty} e^{ikr\left(1 + \frac{i\tau^2}{2}\right)} d\tau =$$

$$= \frac{e^{+\frac{i\pi}{4}} e^{ikr}}{4\pi} \int_{-\infty}^{\infty} e^{-kr\frac{\tau^2}{2}} d\tau =$$

$$= -\frac{e^{ikr + \frac{i\pi}{4}}}{4} \sqrt{\frac{2}{kr\pi}}$$

Asymptotic of $H_0^{(1)}(kr)$

Here we used

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$