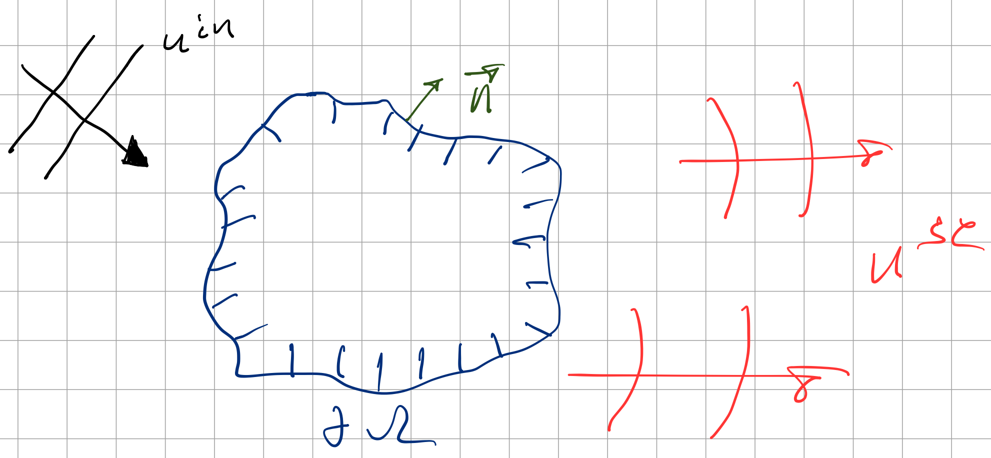
Lecture 2

Problem statement

a) Diffraction by smooth finite obstacles



Governing equation

$$\Delta u + k^2 u = 0$$

Satisfied in Ω

Sometimes we have $\delta(\vec{r} - \vec{r}_s)$ in RHS

By default k is constant, k will become a function for inhomogeneous media.

One can get more complicated equations for media with flow, elastic media, electromagnetic waves, etc.

Plane wave as forcing

Let in Ω u be presented as

$$u = u^{in} + u^{sc}, \quad \text{where}$$

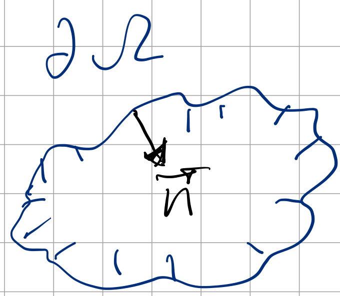
$$u^{in} = e^{-ikr\cos(\varphi - \theta)}$$

We are looking for u^{sc}

Boundary conditions

Neumann

$$\frac{\partial u}{\partial n}(\vec{r}) = 0; \quad \frac{\partial u^{sc}}{\partial n} = -\frac{\partial u^{in}}{\partial n}; \quad \vec{r} \in \partial\Omega$$



Dirichlet

$$u(\vec{r}) = 0; \quad u^{sc} = -u^{in}; \quad \vec{r} \in \partial\Omega$$

Impedance

$$\frac{\partial u}{\partial n} + \beta u = 0;$$

β linked with attenuation or amplification

Boundary conditions can be mixed

Radiation condition

For uniqueness we need to introduce additional condition.

For finite obstacles the Zommerfeld condition is used.

Suppose that u^{sc} behaves as

$$u^{sc}(r,\varphi) \approx B \frac{e^{ikr}}{\sqrt{kr}} S(\varphi), \quad r \gg 1$$

directivity

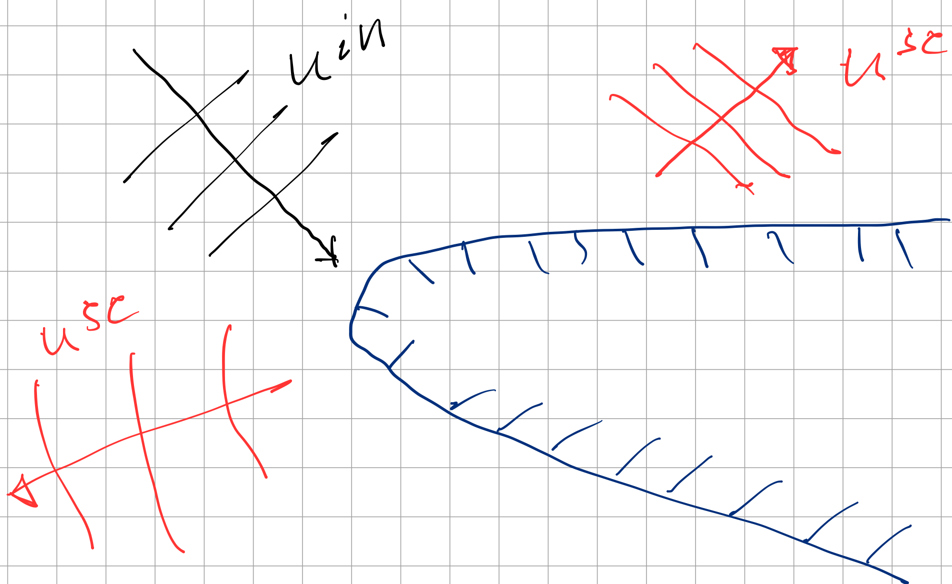
$$B = -\frac{e^{\frac{i\pi}{4}}}{2\sqrt{2\pi}}$$

Zommerfeld condition allows such a wave:

$$\frac{\partial u^{sc}}{\partial r} - iku^{sc} = o\left(\frac{1}{\sqrt{kr}}\right), \quad r \to \infty$$

Limiting absorption principle

Zommerfeld fails for



Alternative:

Let $k = k' + ik''$

k', k'' real positive, $k'' \ll 1$

If time dependence is $e^{-i\omega t}$

k'' corresponds to small

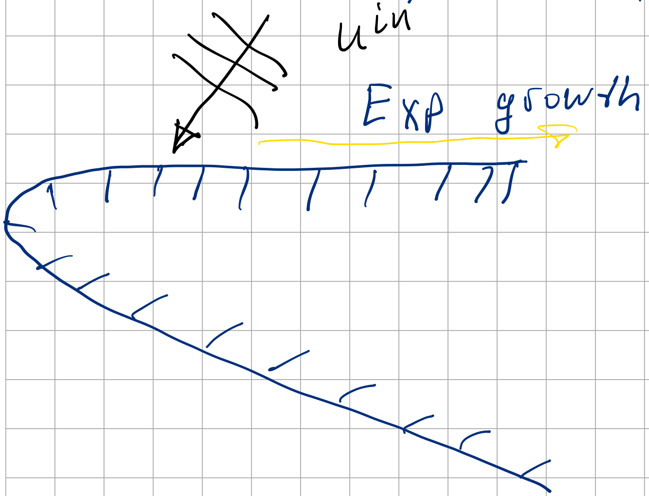
attenuation in the media.
Formally, consider a family of problems for $u^{sc}(x,y,\kappa)$ for $\kappa''>0$, look for exp decaying solutions. Finally, consider a limit

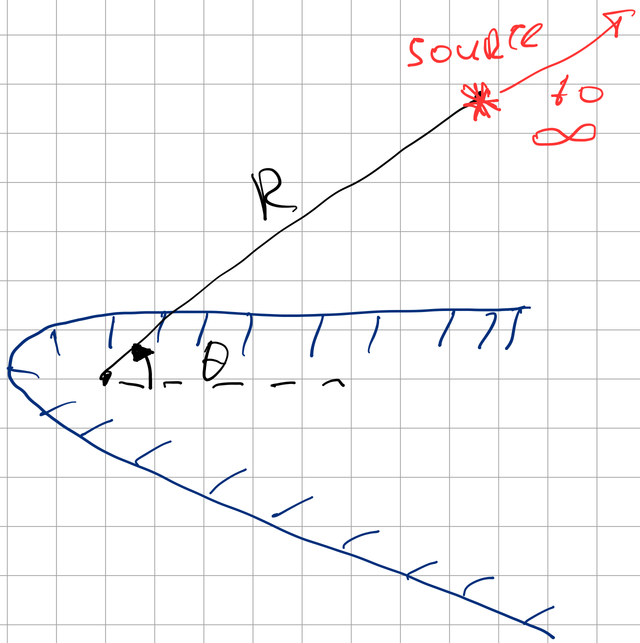
$$u^{sc}(x,y) = \lim_{\kappa''\to 0} u^{sc}(x,y,\kappa)$$

Example: Green's function

$$H_0^{(1)} \sim \frac{e^{i\kappa' r - \kappa'' r}}{\sqrt{\kappa r}}; \quad H_0^{(2)} \sim \frac{e^{-i\kappa' r + \kappa'' r}}{\sqrt{\kappa r}}$$

However, fails:





Safe way is to consider a point source with amplitude $\sqrt{\kappa R}\, e^{i\kappa R}$ and

$$u^{sc}(x,y) = \lim_{R\to\infty} \lim_{\kappa''\to 0} u^{sc}(x,y,\kappa,R)$$

Diffraction by obstacles with edges

Example: Half-plane

$$\Delta u + k^2 u = 0$$

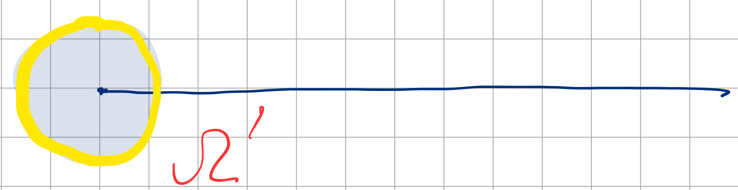
Consider

$$u^{sc} = \frac{e^{ikr}}{\sqrt{kr}} \cos(\varphi/2)$$

solution of the problem for zero u^{in}

We need additional condition to prohibit such sources:

Meixner condition



$$\iint_{\Omega'} |u|^2 \, ds < \infty \qquad \iint_{\Omega'} |\nabla u|^2 \, ds < \infty$$

$$\sim r^{-3/2}$$

does not converge

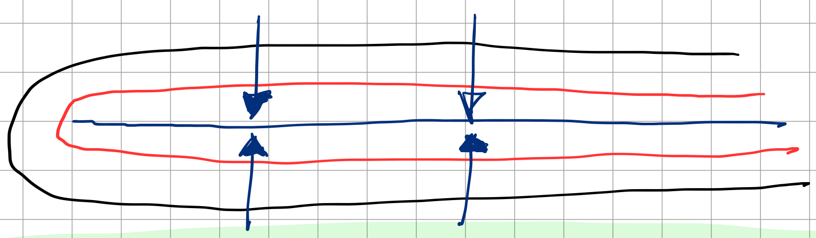
Related concept is Meixner Series (field in the vicinity of the edge)

For Half-plane it is

$$u(r,\varphi) = \sum_{n=0}^{\infty}\left(A_n J_{n/2}(kr) + B_n N_{\frac{n}{2}}(kr)\right)\cos\frac{n\varphi}{2}$$

Meixner condition allows terms with A_n and prohibits with B_n

Formally, one should prove that the solution corresponds to the limit

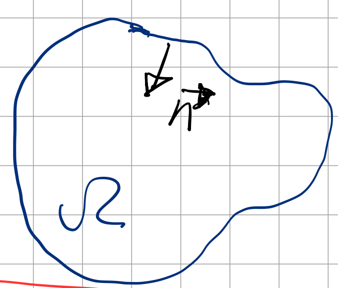


Second Green's identity

Consider a pair of equations

$$\Delta u + k^2 u = f$$

$$\Delta w + k^2 w = g$$



$$\boxed{\iiint_\Omega \left[f w - g u \right] dV =}$$

$$= \iiint_\Omega \left[\Delta u\, w - \Delta w\, u \right] dV =$$

$$= \iiint_\Omega \left[\operatorname{div}\left(\nabla u \cdot w - \nabla w \cdot u \right) - \right.$$

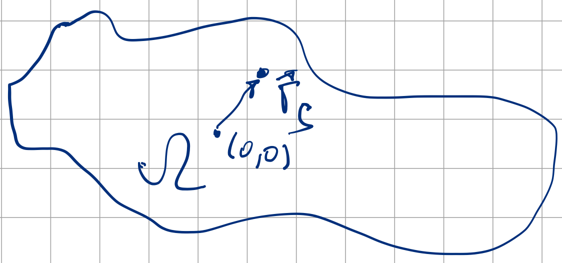
$$\left. - \left(\nabla u \cdot \nabla w - \nabla u \cdot \nabla w \right) \right] dV$$

$$\boxed{= -\iint_{\partial\Omega} \left[\frac{\partial u}{\partial n} \cdot w - \frac{\partial w}{\partial n} \cdot u \right] dS}$$

Applications:

1) Field reconstruction formula

$w(\vec{r}, \vec{r}_s)$ $\qquad g = \delta(\vec{r} - \vec{r}_0)$

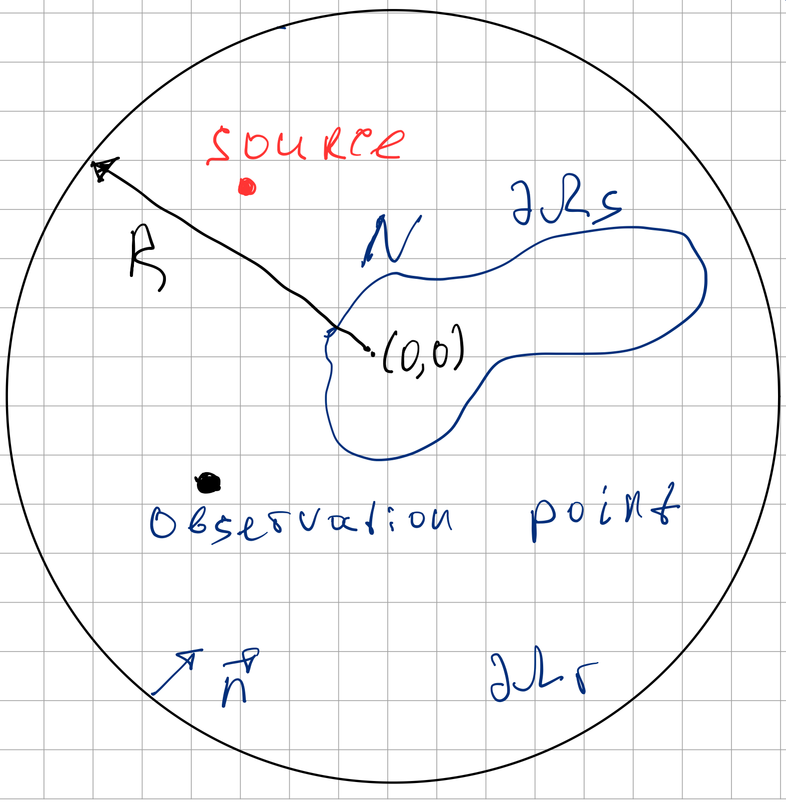


$$-u(\vec{r}_s) = \iint_{\partial\Omega} \left[\frac{\partial u}{\partial n}(r)\, w(\vec{r}, \vec{r}_s) - \frac{\partial w}{\partial n}(\vec{r}, \vec{r}_s)\, u(r) \right] dS$$

Knowing $\frac{\partial u}{\partial n}$ and u on the boundary we can find u everywhere

2) Boundary integral equation

u – solution to diffraction problem



$$f = \delta(\vec{r} - \vec{r}_s)$$

$$W = G(\vec{r}\,', \vec{r}_0)$$

$$g = \delta(\vec{r}\,' - \vec{r}_0)$$

Green's function of free space

2D:

$$W = H_0^{(1)}(k|\vec{r} - \vec{r}_0|)$$

3D:

$$W = -\frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|}$$

$$G(r_s, r_0) - u(r_0) = -\iint_{\partial\Omega_s} \Big[G \underbrace{\frac{\partial u}{\partial n}}_{=0} - u\frac{\partial G}{\partial n}\Big] dS$$

because of Neumann

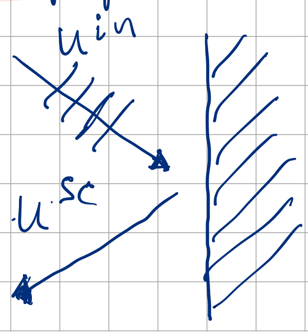
$$- \iint_{\partial\Omega_R} \Big[G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\Big] dS \to 0 \quad R \to \infty$$

$$u(r_0) = G(r_s, r_0) - \iint_{\partial\Omega_s} u(\vec{r})\frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} dS$$

$$\boxed{\frac{1}{2}u(r_0) = G(r_s, r_0) - \iint_{\partial\Omega_s} u(\vec{r})\frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} dS}$$

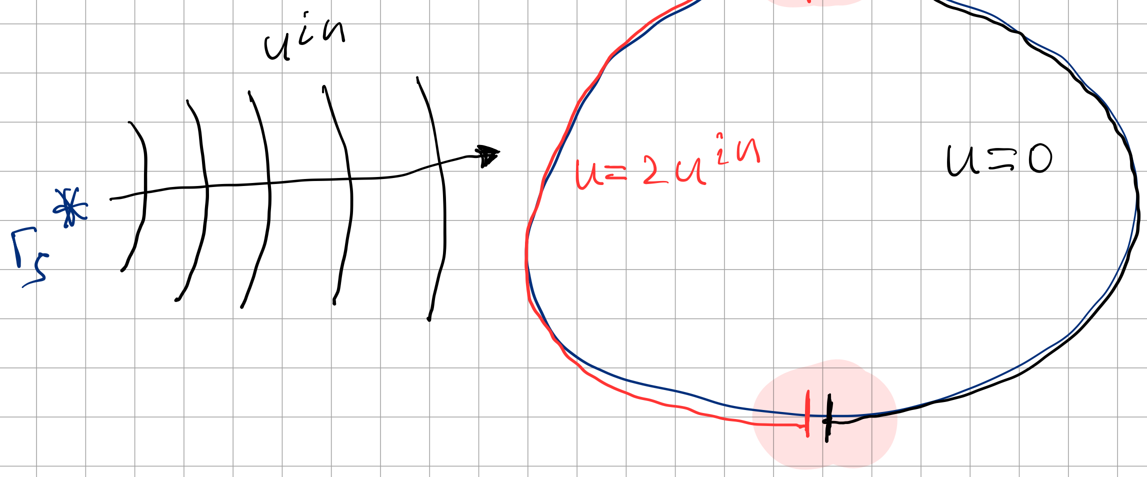
BIE

Kirchoff approximation:



$$u = u^{in} + u^{sc}; \quad u^{sc} = u^{in}$$

$$u_s = 2u^{in}$$

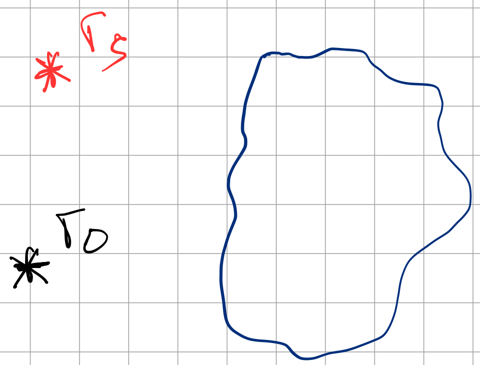


$$u(\vec{r}_0) = G(\vec{r}_s, \vec{r}_0) - 2\iint_{\partial\Omega_s} u^{in}(\vec{r})\,\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n}\,dS$$

Home work:

a) Proove reciprocity theorem

$$u(r_s, r_0) = u(r_0, r_s)$$



b) Directivity as integral over surface of the scatterer