Lecture 3

Geometric theory of diffraction

(Ray approximation)

Let

$$(\Delta + k_0^2 n^2)\, u(x,y) = 0$$

$n = n(x,y) \to$ smooth function

$\downarrow$ index of refraction

Let's look for solution using WKB anzats

$$u(x,y,k_0) = e^{\{i k_0 (G_0(x,y) + k_0^{-1} G_1(x,y) + \dots)\}}$$

as $k_0 \to \infty$

$$\frac{\partial u}{\partial x} = i k_0 \left(\frac{\partial G_0}{\partial x} + k_0^{-1} \frac{\partial G_1}{\partial x} + \dots \right) e^{(\dots)}$$

$$\frac{\partial^2 u}{\partial x^2} = \Big[-k_0^2 \left(\frac{\partial G_0}{\partial x} + k_0^{-1} \frac{\partial G_1}{\partial x} + \dots \right)^2 +$$

$$+ i k_0 \left(\frac{\partial^2 G_0}{\partial x^2} + k_0^{-1} \frac{\partial^2 G_1}{\partial x^2} + \dots \right)$$

$$n^2 - \left(\frac{\partial G_0}{\partial x} \right)^2 - \left(\frac{\partial G_0}{\partial y} \right)^2 = 0$$

$$-2 \left(\frac{\partial G_0}{\partial x} \frac{\partial G_1}{\partial x} + \frac{\partial G_0}{\partial y} \frac{\partial G_1}{\partial y} \right) +$$

$$i \left(\frac{\partial^2 G_0}{\partial x^2} + \frac{\partial^2 G_0}{\partial y^2} \right) = 0$$

$$(*) \qquad \vec{\nabla} G_0 \vec{\nabla} G_0 = n^2 \Rightarrow |\nabla G_0| = n$$

$$(**) \qquad \vec{\nabla} G_0 \cdot \vec{\nabla} G_1 = \frac{i}{2} \Delta G_0$$

(*) – Eikonal equation (phase)

(**) – Transport equation (amplitude)

Another (standard) form

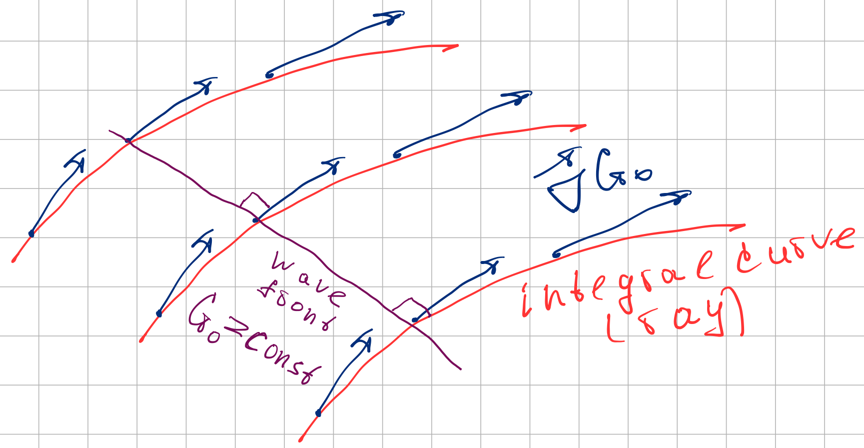
$$u \approx A(x,y)\, e^{i k_0 G_0(x,y)}$$

$$A(x,y) = e^{i G_1(x,y)}$$

$$\nabla G_0 \cdot \nabla A_0 = -\frac{1}{2} A_0 \Delta G_0$$

$\vec{\nabla} G_0$ – Eikonal gradient creates a vector field

Consider an integral curve for $\vec{\nabla} G$:



Thus we can present the wave field as bunch of Rays after finding ∇G

(s, φ) — ray coordinates

$s = G_0$

φ — angle parameter (goes along wavefronts)

$$A_0 \sim \frac{1}{\sqrt{n}}$$

can be found from transport equation

Also follows from conservation of energy

For isotropic media

$$|\nabla G_0| = 1 \qquad G_0 = k_x x + k_y y.$$

plane waves

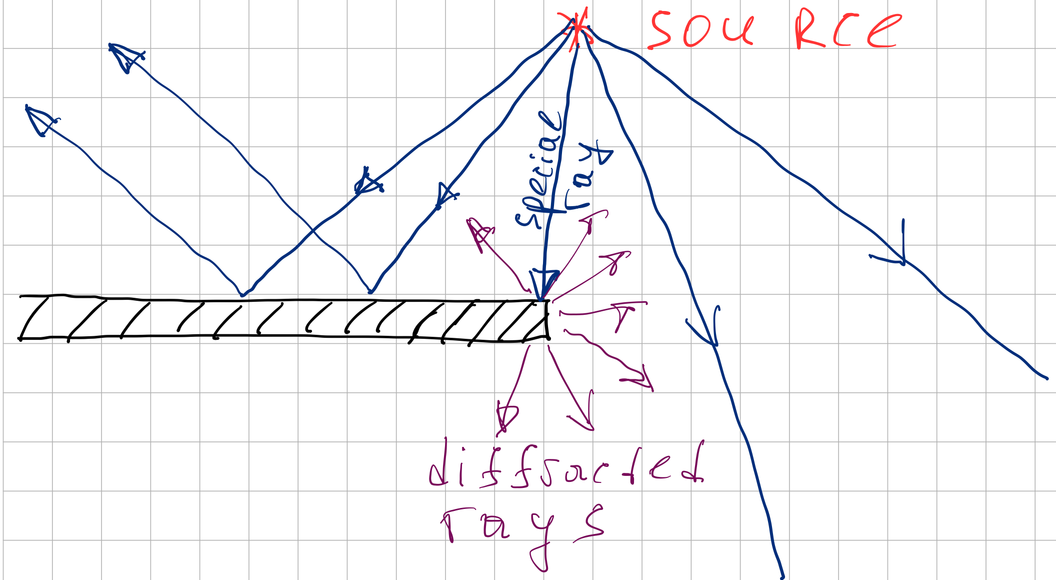
$$A_0 = const$$

Thus, we have GO (geometric optics) postulates:

1. Wavefield travels in straight lines in a homogeneous media
2. Wavefield obeys the law of reflection and refraction

Taking diffraction into account

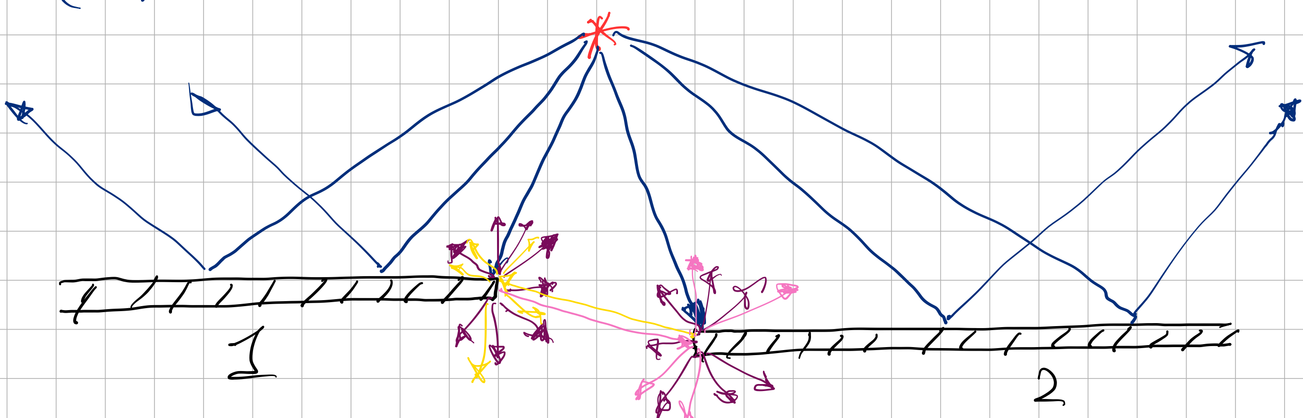
Geometric theory of diffraction (Keller — 1962)



1. Edge acts as a secondary source of diffracted rays
2. Amplitude is determined by diffraction coefficient ≡ directivity

$$u^{diff} = -\frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi}} \frac{e^{ikr_e}}{\sqrt{kr_e}} S(\varphi_e)$$

(r_e, φ_e) — local coordinates



3. Diffraction depends only on the local geometry and properties of the object. Total diffracted field can be calculated as a series of acts of diffraction

$$u^{diff} = u_1 + u_2 + u_{12} + u_{21} + \\ + u_{121} + u_{212} + \dots$$

↓

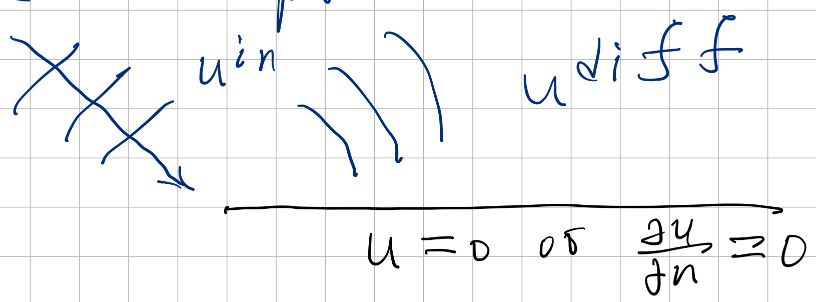
Schwarzschield series

4. Total field computed as sum of all components

$$u = u^{in} + u^{ref} + u^{refr} + u^{diff}$$

Canonical problems of diffraction theory

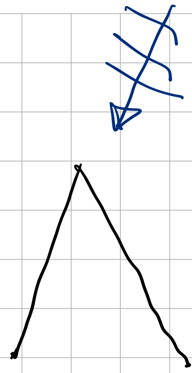
1. Sommerfeld problem.



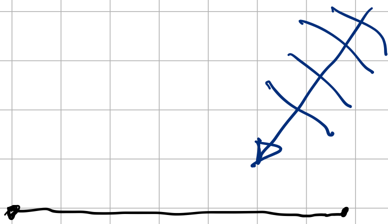
Half-plane or Half-line problem

2. Wedge

u^{in}



3. Strip

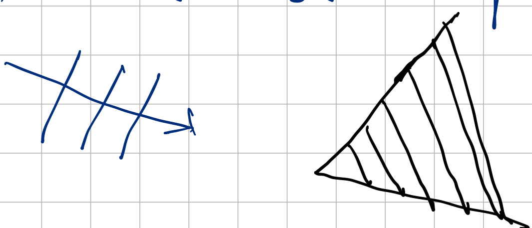


Formally can be solved using Half-plane solution and Diffraction series

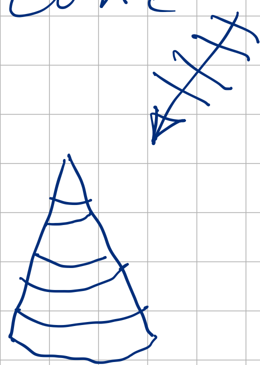
4. Open ended waveguide

3D problems

a) Quarter plane



b) cone



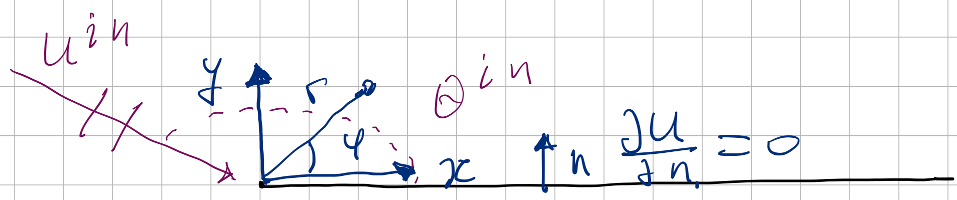
Diffraction by a half-plane

Problem statement

$$\Delta u + k^2 u = 0$$

$$u = u^{in} + u^{sc}$$

$$u^{in} = e^{ikx\cos\theta^{in} + iky\sin\theta^{in}}$$



$$\frac{\partial u}{\partial n} = 0$$

Radiation condition imposed via Limiting absorption principle

Meixner condition

$$u^{sc}(r,\varphi) = \sum_{n=1}^{\infty} \left(A_n J_{n/2}(kr) + B_n N_{\frac{n}{2}}(kr)\right)\cos\frac{n\varphi}{2}$$

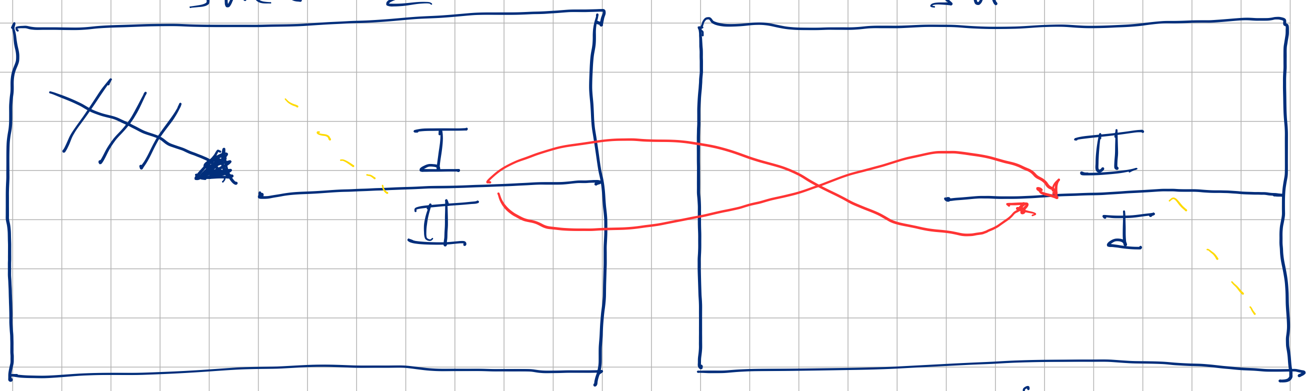
$$J_{\frac{n}{2}}(z) \sim z^{\frac{n}{2}} \qquad z \to 0$$

$$\boxed{u^{sc} \sim \alpha\sqrt{r}\cos\left(\frac{\varphi}{2}\right) + \bar{O}(r)}$$

Solution using Sommerfeld method

a) Reflection principle

sheet 1 sheet 2

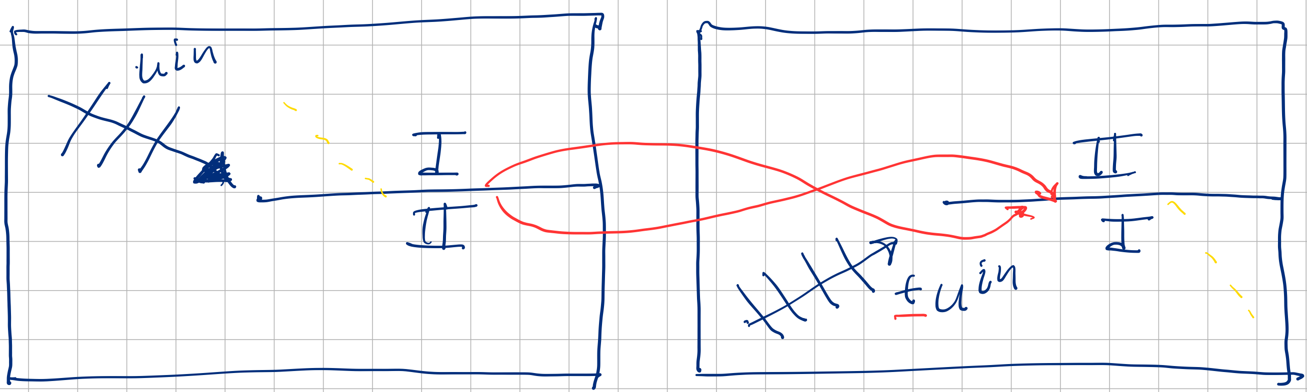


Find a solution $u_z(\theta^{in}; r, \varphi)$ on the Sommerfeld surface

Solution for Neumann problem

$$u = \left(u_z(\theta^{in}, r, \varphi) \pm u_z(4\pi - \theta^{in}, r, \varphi)\right)$$

minus for Dirichlet



By symmetry Neumann conditions will be satisfied at $y=0$ $x>0$

Look for u_z as plane wave decomposition

$$u_z(r, \varphi) = \int_\Gamma S(\theta)\, e^{ikr\cos(\varphi-\theta)}\, d\theta$$

$S(\theta) = 1$ → Green's function