Lecture 4

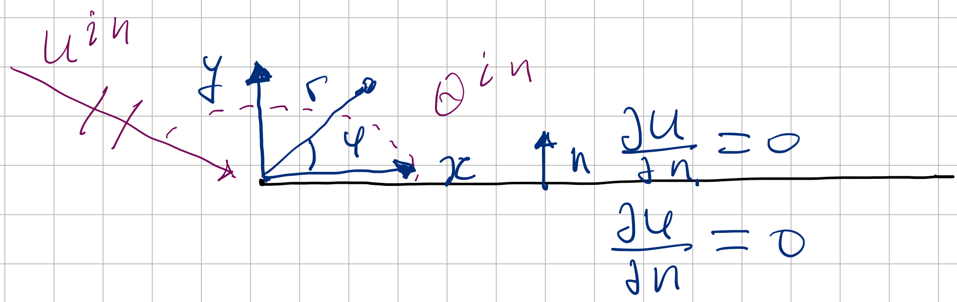
Diffraction by a half-plane

Problem statement

$$\Delta u + k^2 u = 0$$

$$u = u^{in} + u^{sc}$$

$$u^{in} = e^{-ikx\cos\theta^{in} - iky\sin\theta^{in}}$$

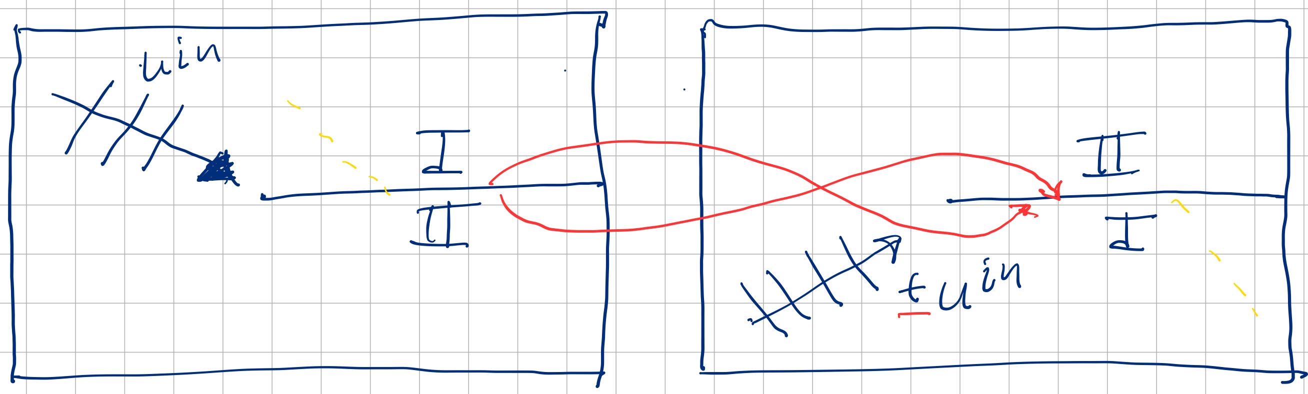


Meixner condition

Radiation condition

$$u = \left(u_2(\theta^{in}, r, \varphi) \pm u_2(4\pi - \theta^{in}, r, \varphi)\right)$$

minus for Dirichlet



By symmetry Neumann conditions will be satisfied at $y=0$ $x>0$

Look for u_z as plane wave decomposition

$$u_z(r, \varphi) = \int_\Gamma S(\theta)\, e^{ikr\cos(\varphi-\theta)}\, d\theta$$

$S(\theta) = 1 \to$ Green's function

Here we stopped last time

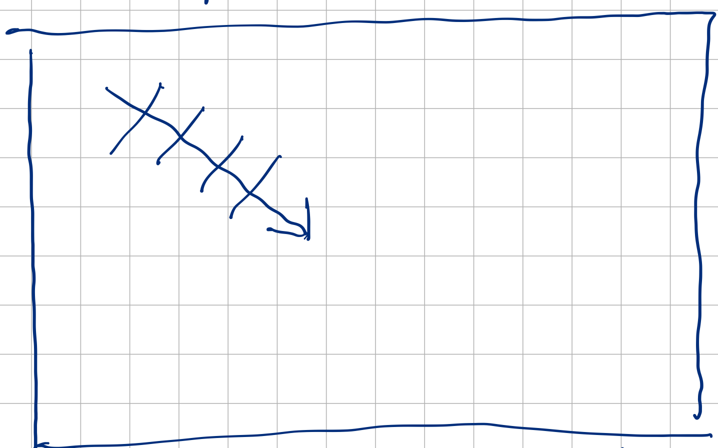
By construction $u_z(r, \varphi)$ is a solution of Helmholtz equation for any reasonable Γ

Let's guess $S(\theta)$

1) $S(\theta)$ should be 4π periodic

2) $S(\theta)$ has a pole on real axis that corresponds to the incident plane wave

Example: Plane wave decomposition of a plane wave



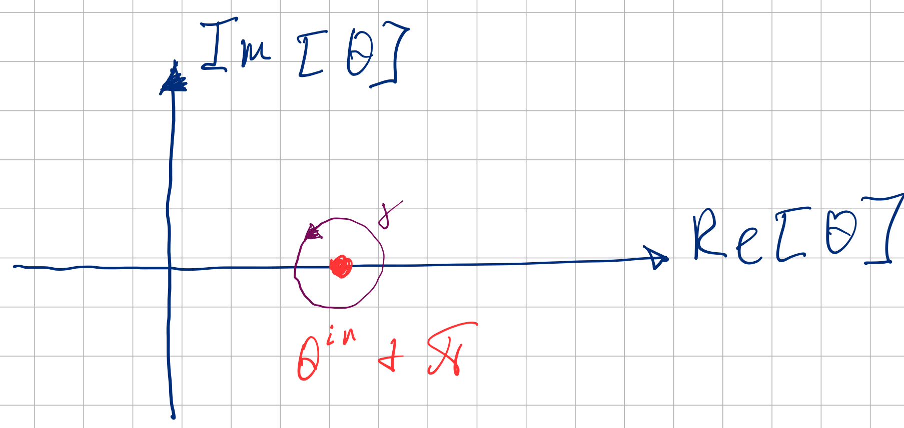
$$u = e^{-ikr\cos(\varphi - \theta^{in})}$$

Equivalently

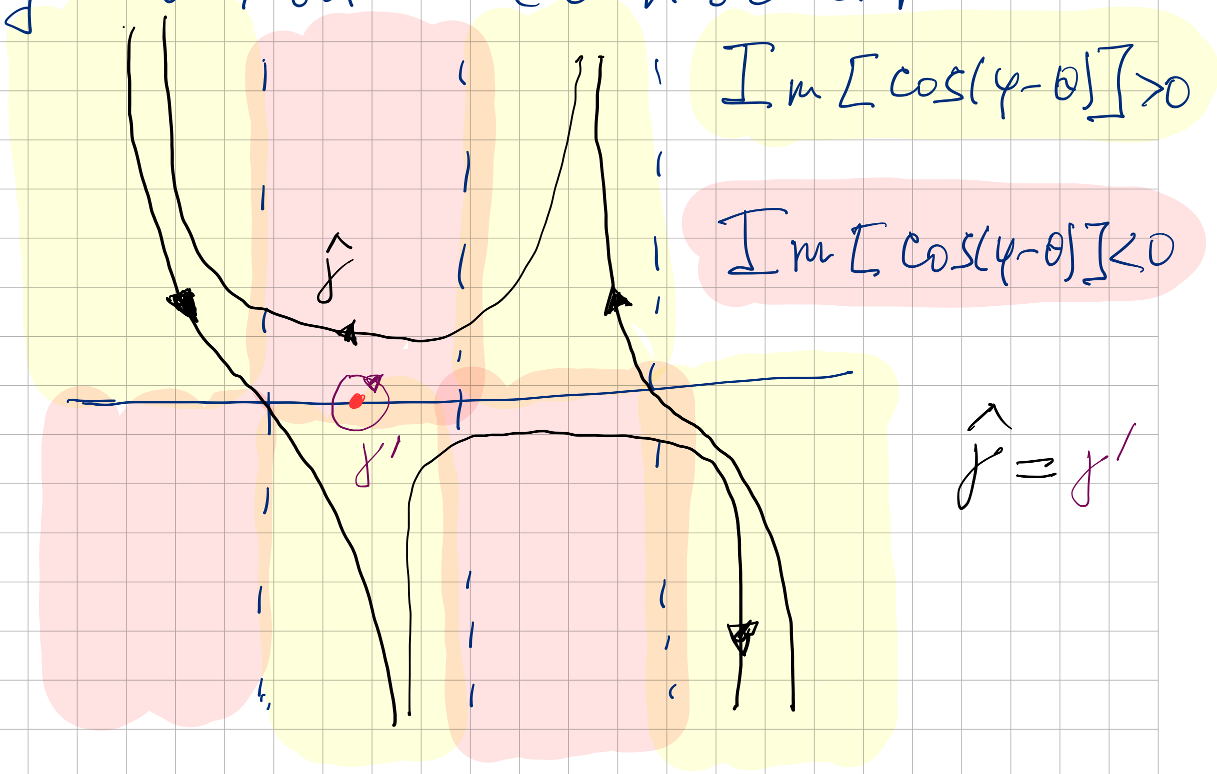
$$u = \int_{\gamma} \frac{1}{2\pi} \frac{e^{i(\theta^{in}+\pi)}}{e^{i\theta} - e^{i(\theta^{in}+\pi)}} e^{ikr\cos(\varphi-\theta)} d\theta =$$

$$\lim_{\theta \to \theta^{in}} \frac{2\pi i}{2\pi} \frac{e^{i(\theta^{in}+\pi)} (\theta - (\theta_{in}+\pi)) e^{ikr\cos(\varphi-\theta)}}{\partial_\theta (e^{i\theta} - e^{i(\theta^{in}+\pi)}) (\theta - (\theta_{in}+\pi))} =$$

$$= e^{ikr\cos(\varphi - \theta^{in})}$$

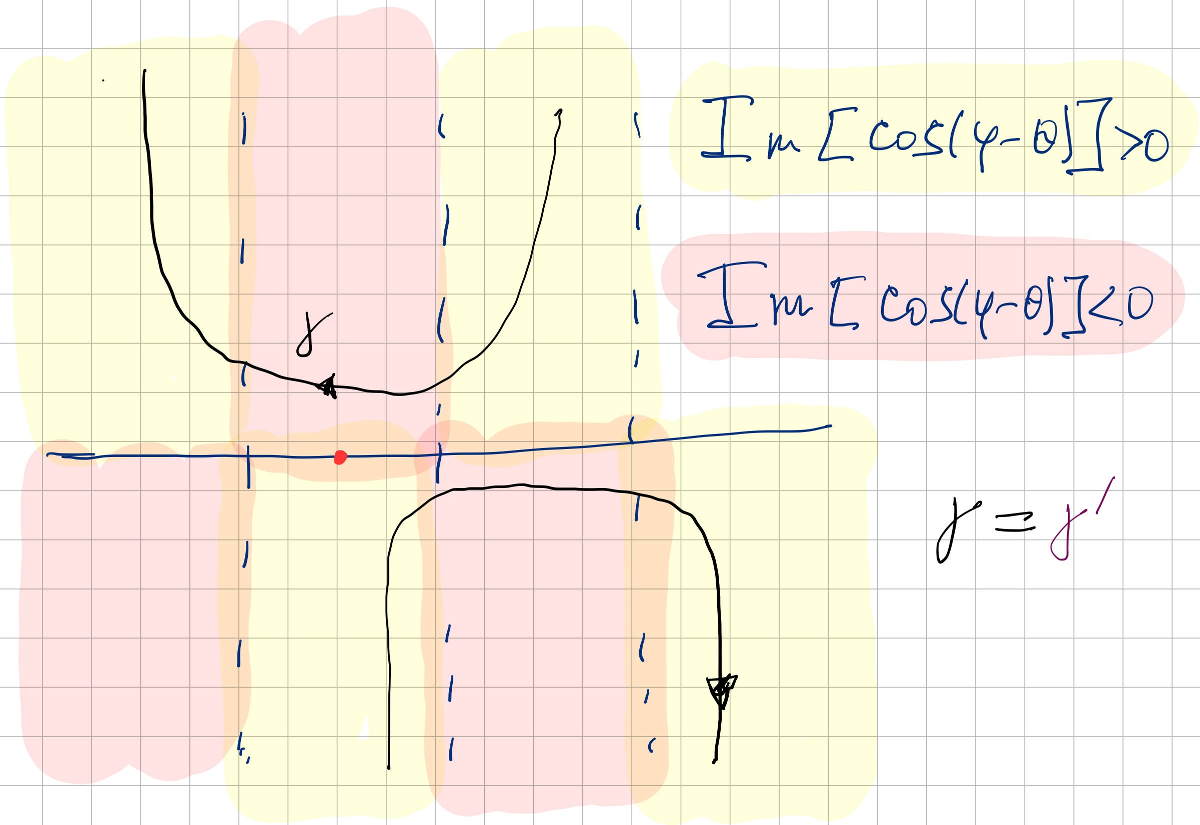


Play with contour



Sommerfeld Genius

Step 1: Reduce the contour due to periodicity



Step 2: Make integrand 4-periodic

$$S(\theta) = \frac{1}{4\pi} \frac{e^{i(\theta^{in}+\pi)/2}}{e^{\frac{i\theta}{2}} - e^{i(\theta^{in}+\pi)/2}}$$

Question: Will

$$\tilde{S}(\theta) = \frac{1}{4\pi} \frac{e^{i\theta}}{e^{i\theta} - e^{i(\theta^{in}+\pi)/2}}$$

give the same result?

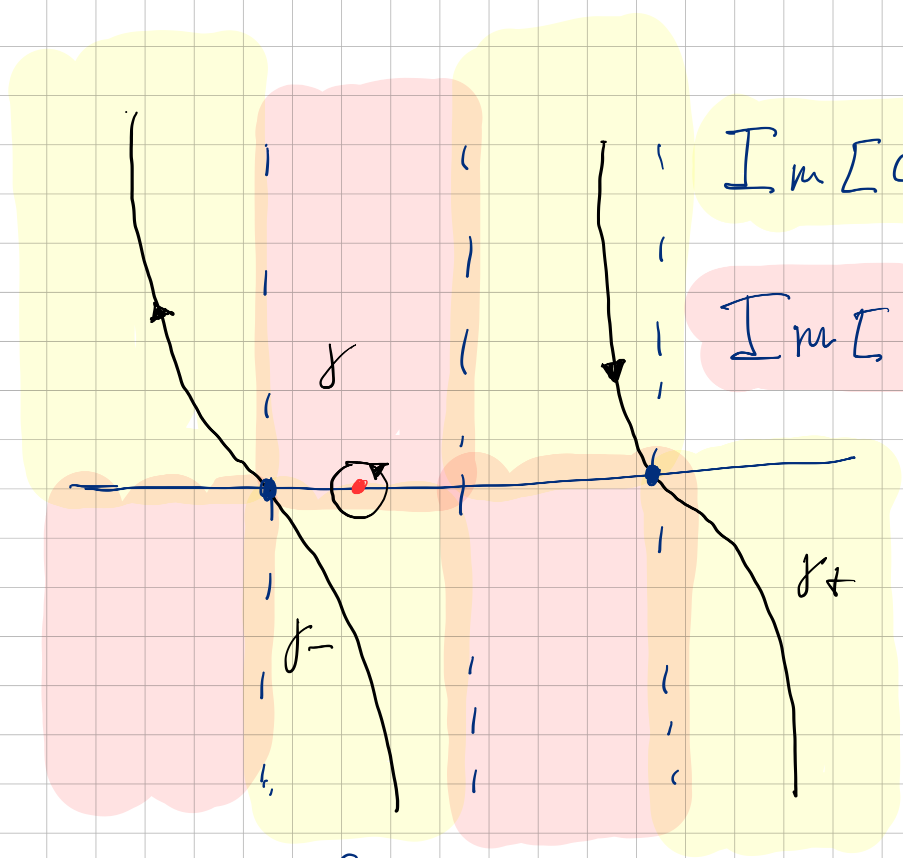
Let's check that what we have is a solution

a) Satisfies Helmholtz equation

$$u_z(r,\varphi) = \int_\gamma S(\theta)\, e^{ikr\cos(\theta-\varphi)}\, d\theta$$

$$(\Delta + k^2)\, e^{ikr\cos(\theta-\varphi)} = 0$$

b) radiation condition



$\hat{\gamma} = \gamma'$

$$u_z(r,\varphi) = \int_{\gamma_-} + \int_{\gamma_+} + u^{in}$$

polar contribution

$$\int_{\gamma_-} S(\theta)\, e^{ikr\cos(\theta-\varphi)}\, d\theta =$$

$$= \int_{\tilde{\gamma}_-} S(\theta+\varphi)\, e^{ikr\cos\theta}\, d\theta \approx$$

$$\approx -S(\varphi) \int_{-\infty}^{\infty} e^{ikr\left(1-\frac{\theta^2}{2}\right)}\, d\theta =$$

$\tilde{\gamma}_- = \gamma_- - \varphi$

$$= S(\varphi)\, e^{ikr - \frac{i\pi}{4}} \sqrt{\frac{2\pi}{kr}}$$

Thus

$$u = D(\varphi)\, \frac{e^{ikr - \frac{i\pi}{4}}}{4\pi} \sqrt{\frac{2\pi}{kr}} + u^{in}$$

$$D(\varphi) = \left(S(2\pi+\varphi) - S(\varphi)\right)(-4\pi)$$

$$D(\varphi) = \frac{1}{\cos\left(\frac{\varphi - \theta^{in}}{2}\right)}$$

$$D_{N,D} = \frac{1}{\cos\left(\frac{\varphi - \theta^{in}}{2}\right)} \underline{\pm} \frac{1}{\cos\left(\frac{\varphi + \theta^{in}}{2}\right)}$$

c) Boundary conditions follow from the symmetry

$$u = \left(u_z(\theta^{in}, r, \varphi) \underline{\pm}\, u_z(4\pi - \theta^{in}, r, \varphi)\right)$$

minus for Dirichlet

d) Meixner condition

Tricky one

Prerequisites

Watson's Lemma

Let $u(x)$ behave as x^β; $x \to 0$

Then, Fourier image

$$U(\xi) = \int_0^\infty u(x)\, e^{-i\xi x}\, dx$$

$$U(\xi) \sim \xi^{-1-\beta} \text{ in}$$

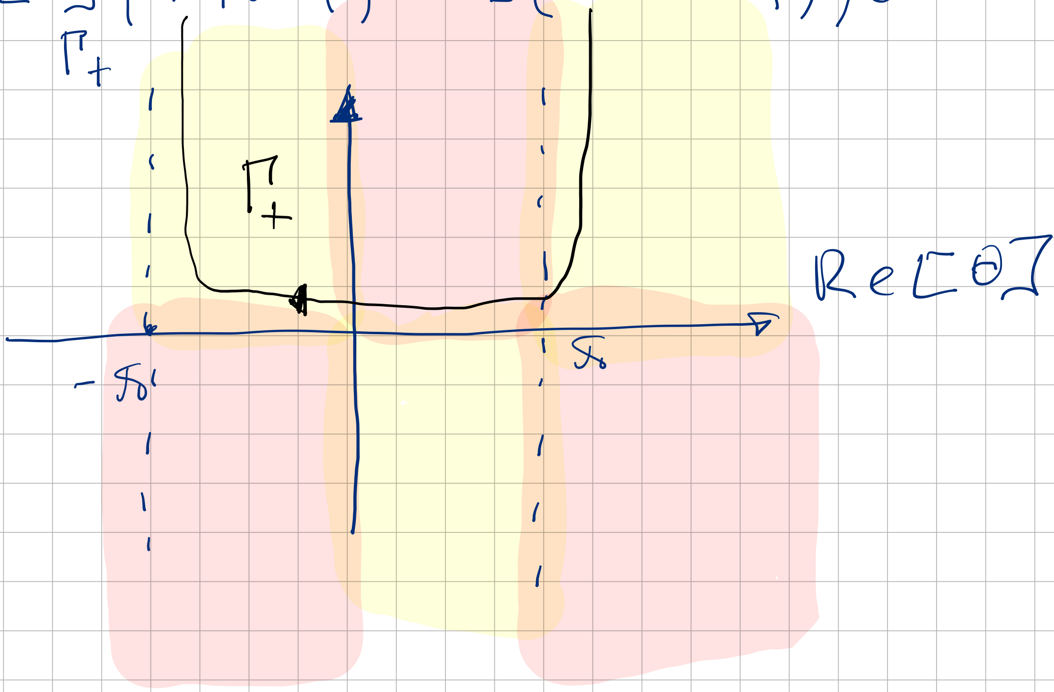
lower half-plane

Half-plane $\beta = \frac{1}{2}$

$$U(\xi) \sim \xi^{-\frac{3}{2}}$$

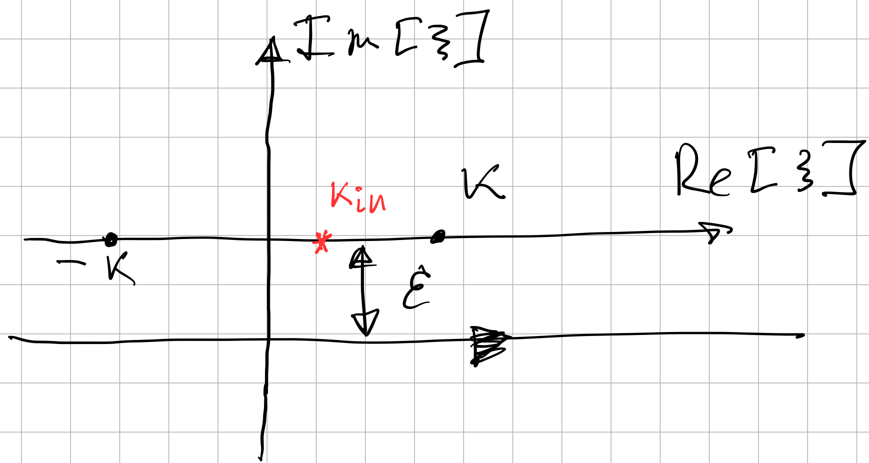
Let's present Sommerfeld integral as Fourier integral

$$u(r,\varphi) = \int_{\Gamma_+} \left(S(\theta+\varphi) - S(2\pi - \theta + \varphi) \right) e^{ikr\cos\theta} d\theta$$



$$\xi = k\cos\theta$$

$$u(r,\varphi) = \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} \frac{S(\theta(\xi)+\varphi) - S(2\pi - \theta(\xi) + \varphi)}{\sqrt{k^2 - \xi^2}} e^{i\xi r} d\xi$$



$g(\xi)$ is a Fourier Transform of a function

$$f(r) = \begin{cases} u(r,\varphi) & , r > 0 \\ 0 & , r < 0 \end{cases}$$

Thus

$$g(\xi) = \int_0^\infty u(r,\varphi)\, e^{-i\xi r}\, d\xi$$

$$g(\xi) \sim \xi^{-3/2}$$

in lower half-plane.

Homework:

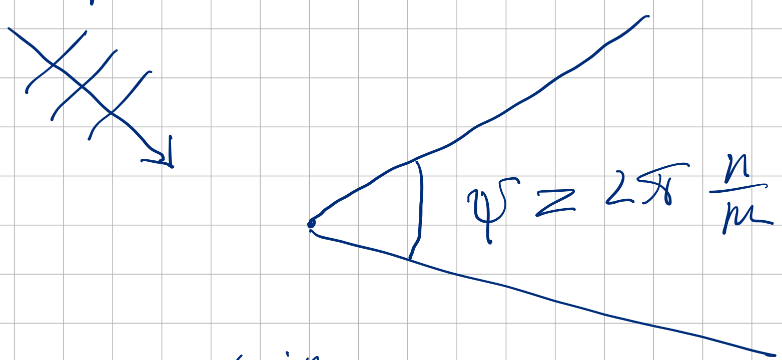
$S(\theta)$ behaves as $e^{\frac{\operatorname{Im}\theta}{2}}$ in LHP

Show that

$g(\xi)$ behaves as $\xi^{-3/2}$ in lower half-plane

Generalizations

Wedge problem



$$S(\theta) = \frac{1}{2\pi n} \frac{e^{\frac{i(\theta^{in}+\pi)}{n}}}{e^{\frac{i\theta}{n}} - e^{\frac{i(\theta^{in}+\pi)}{n}}}$$

Remark: Watson's Lemma

Let $u(x) \sim x^{\beta}$, Study

$$g(\xi) = \int_0^\infty u(x)\, e^{-i\xi x}\, dx$$

Let $Im[\xi] < 0$. Then

$g(\xi)$ regular, and

$$g(\xi) \sim \int_0^\infty x^{\beta} e^{-i\xi x}\, dx + E.S.T.$$

$$\int_0^\infty x^{\beta} e^{-i\xi x}\, dx = \int_0^\infty t^{\beta} e^{-it}\, dt \cdot \left(\frac{1}{\xi^{\beta+1}}\right)$$

$$= C \cdot \xi^{-\beta-1}$$

Thus

$$g(\xi) \sim C \cdot \xi^{-\beta-1}$$