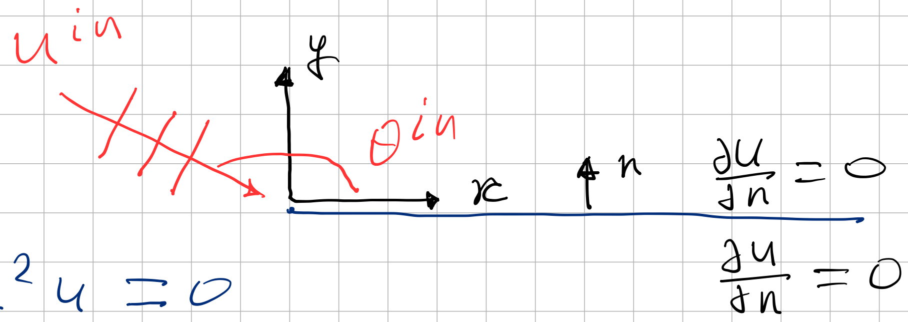
Lecture 5

Diffraction by a half-plane

Wiener-Hopf method

1. Statement. Once again



$$\Delta u + k^2 u = 0$$

$$u = u^{in} + u^{sc}$$

$$u^{in} = e^{-ikx\cos\theta^{in} - iky\sin\theta^{in}}$$

For $\theta^{in} \in \left(\frac{\pi}{2} : \pi\right)$ we can use limiting absorption principle straight away

What changes for other θ^{in}?

Again, statement is completed with Meixner and radiation conditions

Step 1: Symmetrization

Let's state two additional problems: symmetric and antisymmetric:

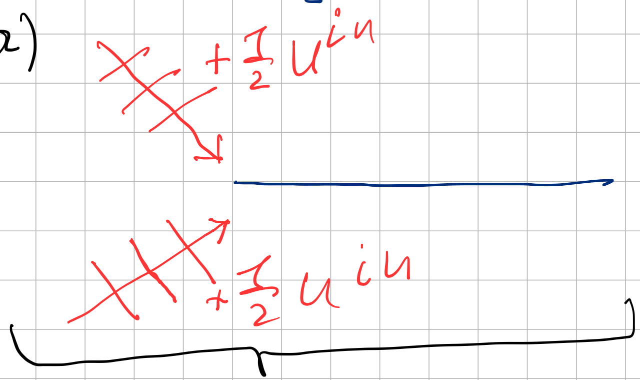
$$u(x,y) = u_s(x,y) + u_a(x,y)$$

$$u_s(x,y) = u_s(x,-y)\ ; \quad u_a(x,y) = -u(x,-y)$$

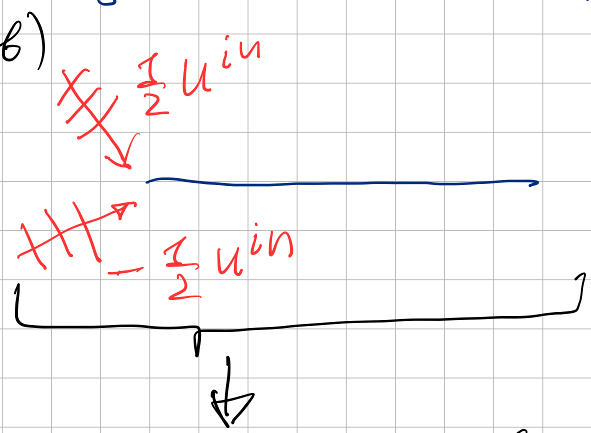
$$u_s^{in}(x,y) = \frac{1}{2}\left(u^{in}(x,y) + u^{in}(x,-y)\right)$$

$$u_a^{in}(x,y) = \frac{1}{2}\left(u^{in}(x,y) - u^{in}(x,-y)\right)$$

a)



b)



Trivial:

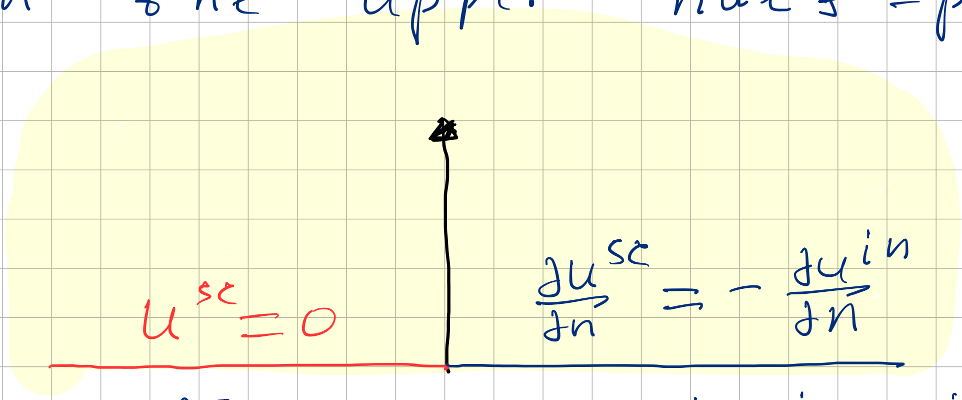
$$\frac{\partial u_s^{in}}{\partial n}(x,0) = 0$$

$$u_s^{sc} = 0$$

Non-trivial

Thus, $u^{sc} = u_a^{sc}$

Now we can state a problem in the upper half-plane:



$$\frac{\partial u^{sc}}{\partial n} = -ik\sin\theta^{in} e^{-ikx\sin\theta^{in}}$$

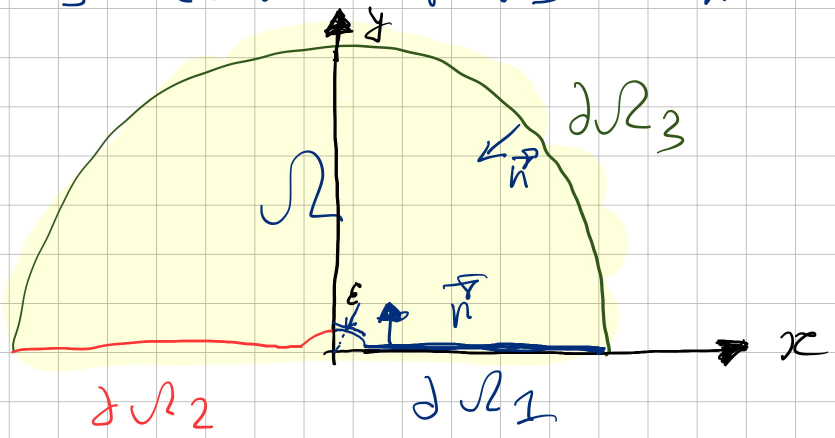
Step 2 Derivation of Wiener-Hopf equations

There are two standard approaches

1. Jones' method (see Noble 1958)

2. Green's formula method

↑ we follow this one



Take a pair

u^{sc} and $w(x,y,\xi) = e^{i\xi x + i\sqrt{k^2-\xi^2}\,y}$

For $|\xi| < k$ let $\sqrt{}$ be close to positive real

For $|\xi| > k$ let $\sqrt{}$ be close to positive imaginary

$$\int_{\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3} \left[u^{sc}\frac{\partial w}{\partial n} - w\frac{\partial u^{sc}}{\partial n}\right] dl = 0$$

$$\int_{\partial\Omega_3} \left[u^{sc}\frac{\partial w}{\partial n} - w\frac{\partial u^{sc}}{\partial n}\right] dl \to 0$$

due to limiting absorption principle

Radius ε of the small circle can be taken to zero due to Meixner condition

Thus

$$\int_{\partial\Omega_1 \cup \partial\Omega_2} \left[u^{sc} \frac{\partial w}{\partial n} - w \frac{\partial u^{sc}}{\partial n} \right] = 0$$

$$\parallel$$

$$\int_{-\infty}^{\infty}$$

$$\frac{\partial w}{\partial n} = \left.\frac{\partial w}{\partial y}\right|_{x=0} = i\sqrt{k^2-\xi^2}\, e^{i\xi x}$$

$$w(x,0) = e^{ikx}$$

$$-\int_{-\infty}^{0} \frac{\partial u^{sc}}{\partial y} e^{i\xi x} + i\sqrt{k^2-\xi^2} \int_0^{\infty} u^{sc} e^{i\xi x} +$$

$$+ \int_0^{\infty} e^{i\xi x - ikx\cos\theta^{in}} (+ik\sin\theta^{in})$$

$$\parallel \quad \frac{-k\sin\theta^{in}}{\xi - k\cos\theta^{in}}$$

Finally, Wiener - Hopf equation:

$$W_-(\xi) - i\sqrt{k^2-\xi^2}\, U_+(\xi) = \frac{-k\sin\theta^{in}}{\xi - k\cos\theta^{in}}$$

$$W_-(\xi) = \int_{-\infty}^{0} \frac{\partial u^{sc}}{\partial y}(x,0)\, e^{i\xi x}\, dx$$

$\downarrow$ analytic in LHP

$$U_+(\xi) = \int_0^{\infty} u^{sc}(x,0)\, e^{i\xi x}\, dx$$

$\downarrow$ analytic in LHP

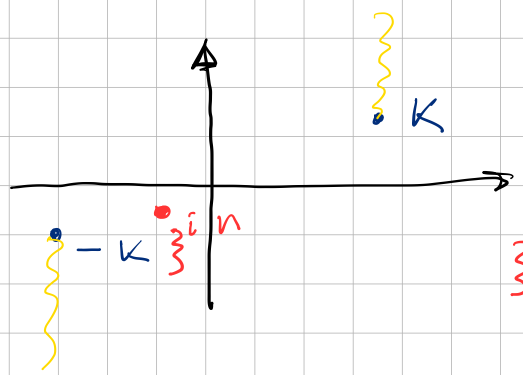
From Watson Lemma

$W_-(\xi) \sim \xi^{-3/2}$ as $\xi \to \infty$ in LHP

$U_+(\xi) \sim \xi^{-\frac{1}{2}}$ as $\xi \to \infty$ in UHP

Kernel:

$K(\xi) = -i\sqrt{k^2-\xi^2}$ has 2 branch points



$\xi^{in} = k\cos\theta^{in}$

RHS: $R(\xi) = \dfrac{-k\sin\theta^{in}}{\xi - k\cos\theta^{in}}$

Note that $\cos\theta^{in} < 0$. Thus $R(\xi)$ analytic in UHP and has a pole in LHP

Step 3 Solution to W-H equation

a) Factorisation of the Kernel

$$K(\xi) = K_+(\xi)\, K_-(\xi)$$

analytic and don't have zeros in LHP

analytic and don't have zeros in UHP

$K_-(\xi) = \sqrt{k-\xi}$

$K_+(\xi) = -i\sqrt{k+\xi}$

Divide W-H equation by $K_-(\xi)$

$$\frac{W_-(\xi)}{\sqrt{k-\xi}} - i\sqrt{k+\xi}\; U_+(\xi) = \frac{-k\sin\theta^{in}}{(\xi-\xi^{in})\sqrt{k-\xi}}$$

b) additive Factorization of RHS

$$R'(\xi) = \frac{-k \sin\theta^{in}}{(\xi - \xi^{in})\sqrt{k-\xi}}$$

$$R'(\xi) = R'_+(\xi) + R'_-(\xi)$$

Done via pole removal:

$$R'_-(\xi) = \frac{-k\sin\theta^{in}}{(\xi-\xi^{in})\sqrt{k-\xi}} + \frac{k\sin\theta^{in}}{(\xi-\xi^{in})\sqrt{k-\xi^{in}}}$$

$$R'_+(\xi) = -\frac{k\sin\theta^{in}}{(\xi-\xi^{in})\sqrt{k-\xi^{in}}}$$

c) Application of Liouville's theorem

$$F(\xi) = \underbrace{\frac{W_-(\xi)}{K_-(\xi)} - R'_-(\xi)}_{\text{decays as } \xi^{-1}} = \underbrace{-K_+(\xi)\,U_+(\xi) + R'_+(\xi)}_{\text{decays as } \xi}$$

$F(\xi)$ holomorphic everywhere and decays at ∞

Thus

$$F(\xi) = 0$$

$$\boxed{U_+(\xi)} = \frac{R'_+(\xi)}{K_+(\xi)} = \frac{-i k \sin\theta^{in}}{(\xi-\xi^{in})\sqrt{k-\xi^{in}}\sqrt{k+\xi}}$$

$\downarrow$

Solution

Indeed, we found

u^{sc} and $\frac{\partial u^{sc}}{\partial n}$ on the half-plane

$u^{sc} = \mathcal{F}^{-1}[U_d(\xi)]$

Using Green's reconstruction formula we can find the field everywhere:

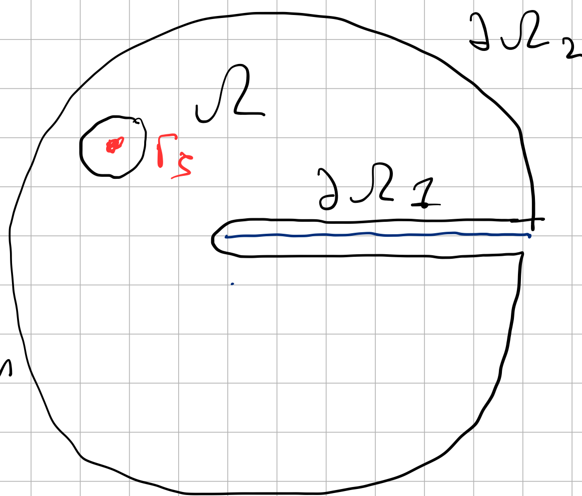
$$u^{sc}(x,y) = \frac{ik\sin\theta^{in}}{2\pi\sqrt{k-\xi_p}} \int_{-\infty}^{\infty} \frac{e^{i\xi x + i\sqrt{k^2-\xi^2}\,y}}{(\xi+\xi^{in})\sqrt{k-\xi}}\, d\xi$$

Homework:

a) Obtain the latter yourself

Tip: Apply Green's theorem in

Take a pair u^{sc} and $G(r-r_s)$, use plane wave decomposition for $G(r-r_s)$



b) Proove that it is equivalent to the solution via Sommerfeld method

c) Estimate $u^{sc}(x,y)$ as $r=\sqrt{x^2+y^2}$ tends to ∞ using saddle point method