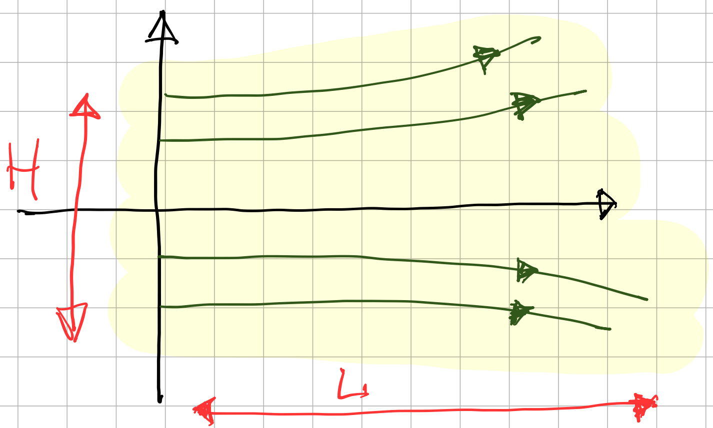
Lecture 6.

1. Parabolic equation of diffraction theory

Consider Helmholtz equation

$$\Delta \tilde{u} + k^2 \tilde{u} = 0$$

Let u be a paraxial wave beam moving along x-axis



How to describe such a beam?

$$k^{-1} \ll H \ll L$$

High frequency case

Let

$$\tilde{u}(x,y) = e^{ikx}\, U(x,y)$$

amplitude function

$$\frac{\partial U}{\partial x} \sim \frac{U}{L}, \quad \frac{\partial U}{\partial y} \sim \frac{U}{H}$$

$$\Big(\underbrace{\frac{\partial^2}{\partial y^2}}_{H^{-2}} + \underbrace{2ik\frac{\partial}{\partial x}}_{kL^{-1}} + \underbrace{\frac{\partial^2}{\partial x^2}}_{L^{-2}}\Big) U(x,y) = 0$$

$$\left(\frac{\partial^2}{\partial y^2} + 2ik\frac{\partial}{\partial x}\right)U(x,y) = 0$$

$$H \sim \sqrt{\frac{L}{k}}$$

a) due to $\frac{\partial}{\partial x}$ waves propagate from left to right

Consider:

$$\tilde{u}^{in} = e^{ikx\cos\theta + iky\sin\theta} \simeq$$

$$\simeq |\theta \ll 1| \simeq e^{ikx} e^{-ikx\frac{\theta^2}{2} + iky\theta}$$

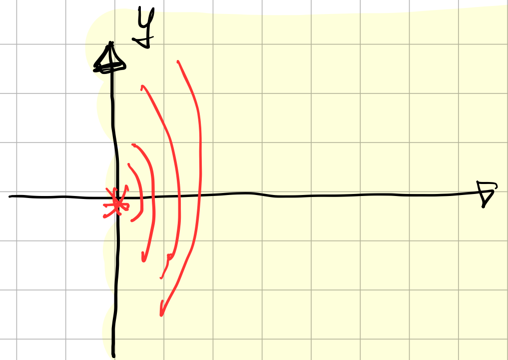
$$u^{in} = e^{-ikx\frac{\theta^2}{2} + iky\theta}$$ → parabolic plane wave

2. Green's function of the parabolic equation

$$\left(\frac{\partial^2}{\partial y^2} + 2ik\frac{\partial}{\partial x}\right)U(x,y) = 0$$

Let's solve it for $x > 0$ with the boundary (initial) condition

$$U(0,y) = \delta(y)$$



Denote the solution

$$U(x,y) = G(x,y)$$

Let's look for the solution as plane wave decomposition

$$G(x,y) = \int_{-\infty}^{\infty} A(\xi)\, e^{i\xi y - i\xi^2 x/(2K)}\, d\xi$$

From boundary condition

$$\int_{-\infty}^{\infty} A(\xi)\, e^{i\xi y}\, d\xi = \delta(y),$$

i.e. $A(\xi) = \frac{1}{2\pi}$

$$G(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{ \underbrace{i\xi y - \frac{i\xi^2 x}{2K}}_{} \right\} d\xi =$$

$$= \left| \xi y - \frac{\xi^2 x}{2K} = -\frac{x}{2K}\left(\xi - \frac{yK}{x}\right)^2 + \frac{Ky^2}{2x} \right| =$$

$$= \frac{1}{2\pi} e^{\frac{iKy^2}{2x}} \int_{-\infty}^{\infty} e^{-\frac{ix}{2K}\tau^2}\, d\tau =$$

$$\boxed{\int_{-\infty}^{\infty} e^{-ia\tau^2} = \sqrt{\pi}\, e^{-\frac{i\pi}{4}}\, a^{-\frac{1}{2}}}$$

$$= \sqrt{\frac{K}{2\pi x}}\, e^{\frac{iKy^2}{2x} - \frac{i\pi}{4}}$$

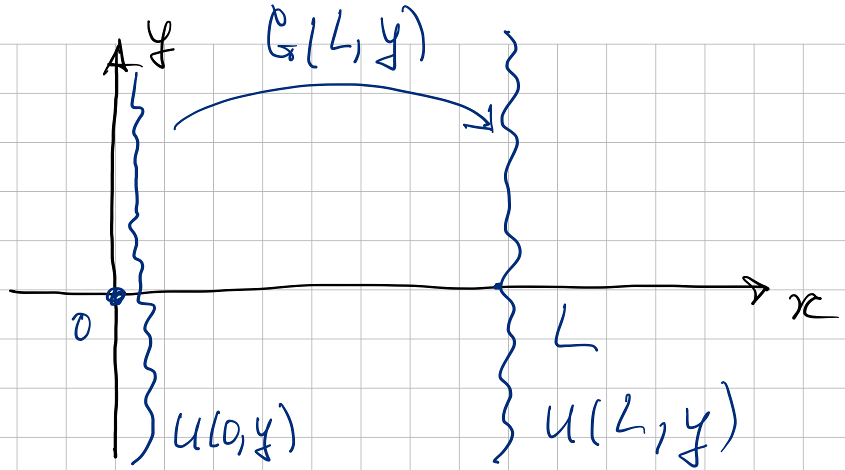
$G(x,y)$ acts as a propagator:

Let $U(0,y) = f(y)$, Then

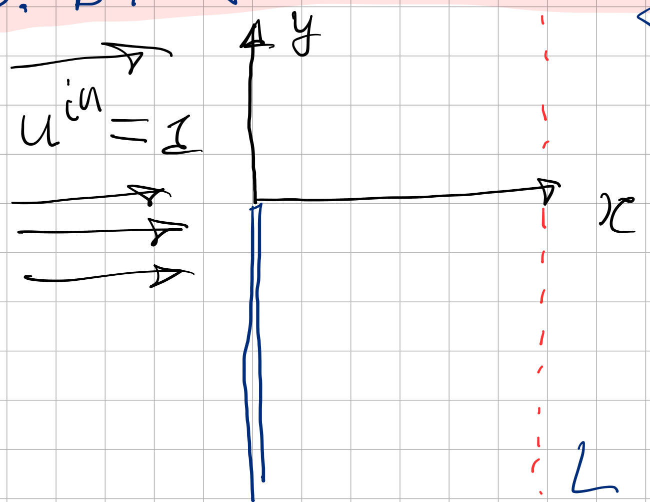
$$f(y) = \int_{-\infty}^{\infty} \delta(y-y')\, f(y')\, dy'$$

Thus

$$U(x,y) = \int_{-\infty}^{\infty} G(x, y-y')\, f(y')\, dy'$$



3. Diffraction by a half-plane
(perpendicular)



$$u(L,y) = \sqrt{\frac{k}{2\pi L}} \int_0^{\infty} e^{\frac{ik(y-y')^2}{2L} - \frac{i\pi}{4}} dy'$$

can be expressed in terms of Erf

Lets estimate asymptotically

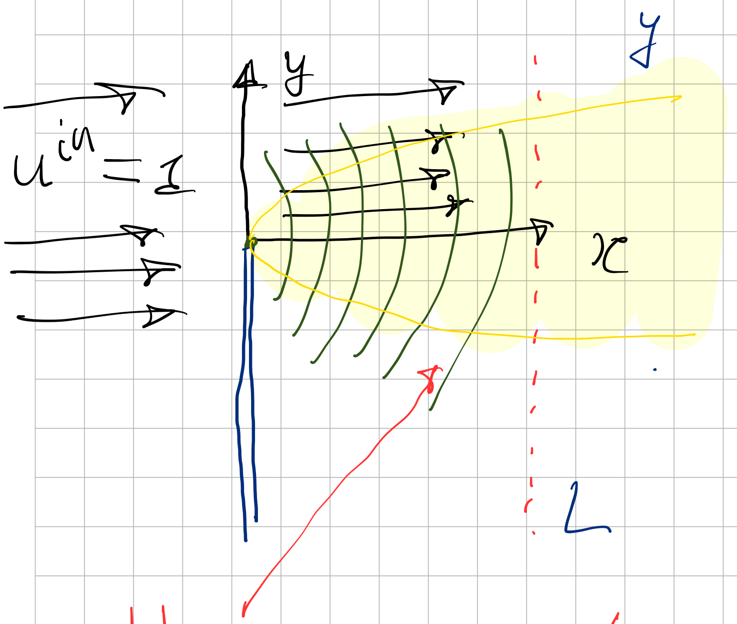
$$\tau = \sqrt{\frac{k}{2L}}(y'-y)$$

$$u(L,y) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \int_{-\sqrt{\frac{k}{2L}}y}^{\infty} e^{i\tau^2} d\tau$$

For $y > 0$ we have a stationary point $\tau = 0$, thus

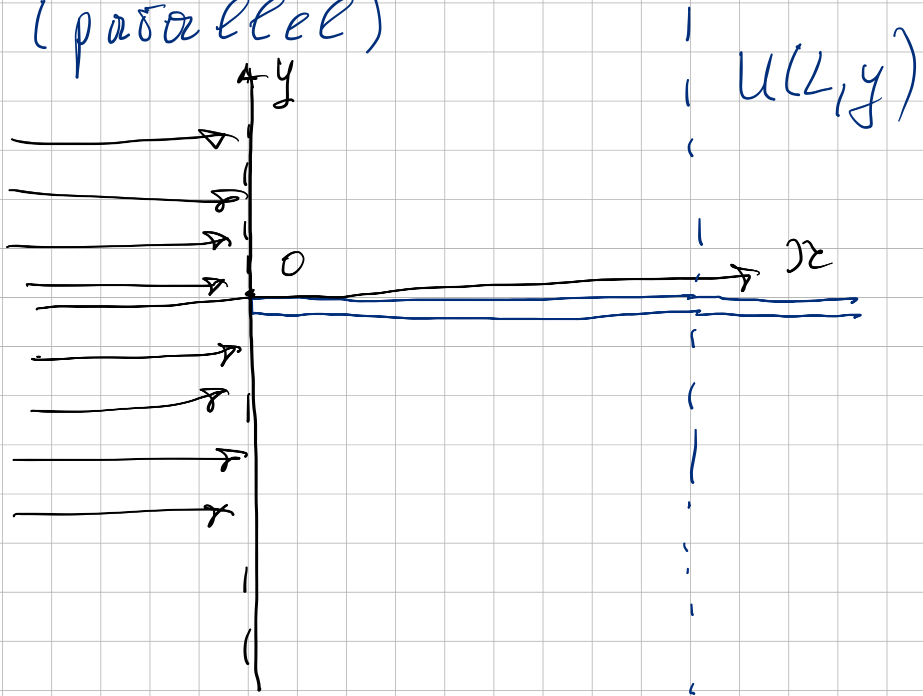
$$U(L,y) \approx \begin{cases} 1 + \dfrac{e^{+\frac{i\pi}{4}}}{\sqrt{\pi}} \dfrac{e^{\frac{iky^2}{2L}}}{2y\sqrt{\frac{k}{2L}}} ; & y>0 \\ + \dfrac{e^{+\frac{i\pi}{4}}}{\sqrt{\pi}} \dfrac{e^{\frac{iky^2}{2L}}}{2y\sqrt{\frac{k}{2L}}} ; & y<0 \end{cases}$$

This term can be obtained by integration by parts



Here the result will be close to the exact solution

4. Diffraction by a half-plane (parallel)



Similarly

$$U(L,y) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \int_{-\sqrt{\frac{k}{2L}}|y|}^{\infty} e^{i\tau^2}\, d\tau$$

wrong see comment in Lecture 7!

Remark.

$$u^{sc} \approx G(x,y)\, S(\theta) \;;$$

$$S(\theta) = \frac{i}{k\theta}, \text{ where } \quad \theta = \frac{y}{L}$$

5. Directivity in the parabolic approximation

$$u^{sc}(x,y) = S(\theta) \cdot G(x,y) + O\left((kx)^{-3/2}\right)$$

$\theta = \frac{y}{x}$, i.e θ is tan of the angle of observation

We use PEDT when $\tan\theta \sim \theta$

Let u^{sc} be known on a line

Using propagator

$$u^{sc}(x,y) = \int_{-\infty}^{\infty} u^{sc}(0,y')\, G(x, y-y')\, dy' =$$

$$= \sqrt{\frac{k}{2\pi x}}\, e^{-\frac{i\pi}{4}} \int_{-\infty}^{\infty} u^{sc}(0,y')\, e^{\frac{ik(y-y')^2}{2x}}\, dy' =$$

$$= \sqrt{\frac{k}{2\pi x}}\, e^{-\frac{i\pi}{4}}\, e^{\frac{iky^2}{2x}} \int_{-\infty}^{\infty} u^{sc}(0,y')\, e^{-\frac{ikyy'}{x} + \frac{ik(y')^2}{2x}} dy'$$

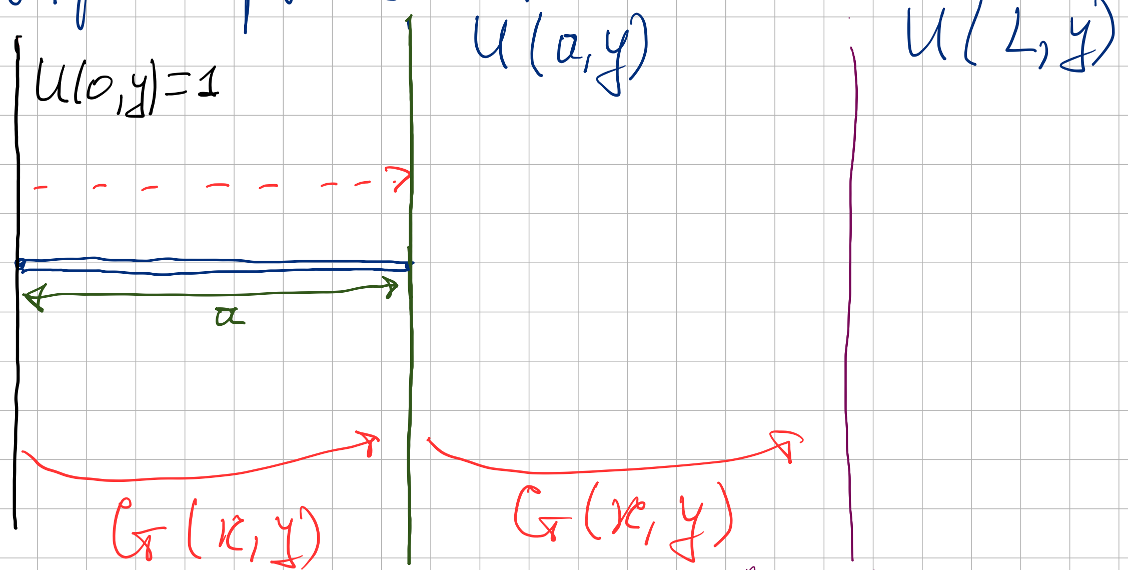
Let $x, y \to \infty$ with $\theta = \frac{y}{x} = \text{const}$

Then:

$$\boxed{S(\theta) = \int_{-\infty}^{\infty} u^{sc}(0,y)\, e^{-iky\theta}\, dy}$$

Homework.

Strip problem



Korol'kov A.I., Shanin A.V. Acoust. Journal V. 62 p. 399 (2016)

Formally $u(L,y) = \Pi_1 \Pi_2 u(0,y)$

Bonus: Asymptotic of Fresnel integral

$$F(z) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \int_z^{\infty} 2i\tau\, e^{i\tau^2} \frac{d\tau}{2i\tau} \quad ; \quad z > 0$$

$$F(z) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \frac{e^{i\tau^2}}{2i\tau}\Big|_z^{\infty} + \int_z^{\infty} \frac{e^{i\tau^2}}{2i\tau^2} d\tau \simeq$$

$$\simeq + \frac{e^{+\frac{i\pi}{4}}}{2\sqrt{\pi}} e^{iz^2} \left(\frac{1}{z} + O\left(\frac{1}{z^2}\right) \right)$$