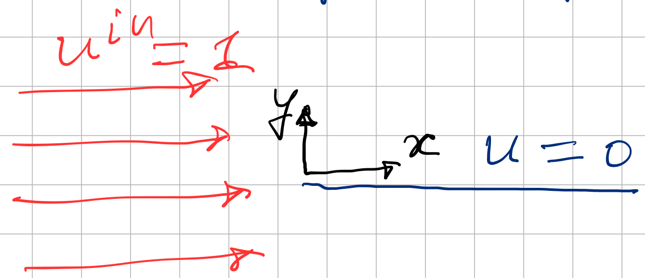
Lecture 7

Parabolic equation in polar coordinates

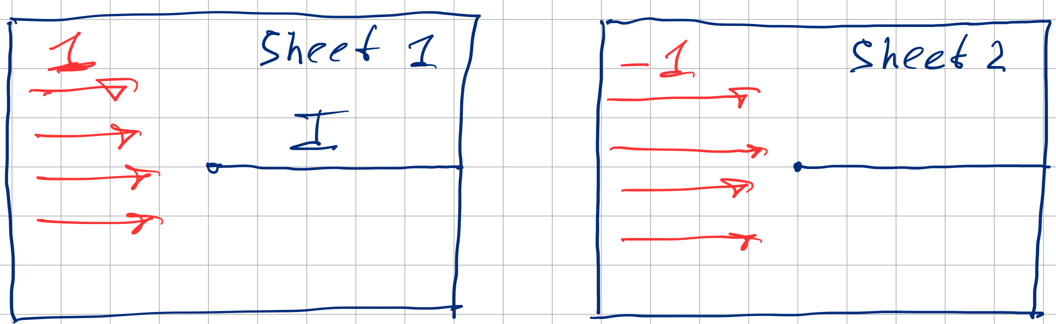
1. Comment

Problem of diffraction by half-plane (parallel)

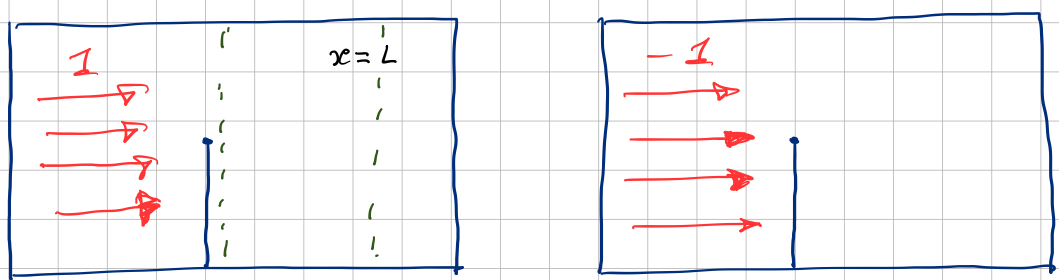


Correct solution:

a) Apply method of reflections



a') Deform the cut



b) Apply propagator formula on sheet 1:

$$u(L,y) = \sqrt{\frac{k}{2\pi L}} \int_0^{\infty} e^{\frac{ik(y-y')^2}{2L} - \frac{i\pi}{4}} dy' -$$

$$- \sqrt{\frac{k}{2\pi L}} \int_{-\infty}^{0} e^{\frac{ik(y-y')^2}{2L} - \frac{i\pi}{4}} dy' =$$

$$= \mathcal{F}\left(-\sqrt{\frac{k}{2L}}\, y\right) - \mathcal{F}\left(\sqrt{\frac{k}{2L}}\, y\right),$$

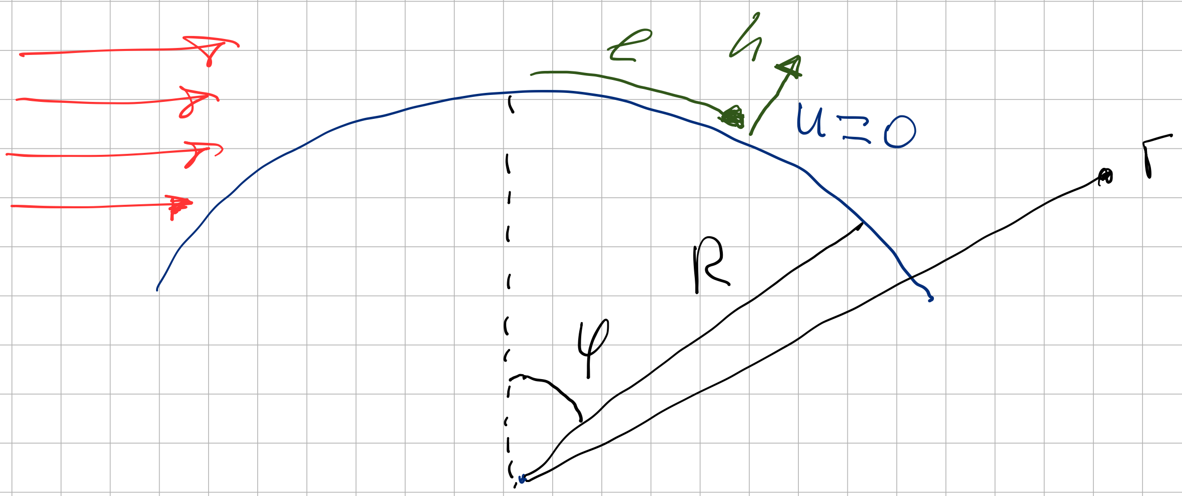
where

$$\mathcal{F}(z) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \int_z^{\infty} e^{i\tau^2} d\tau,$$

i.e. on Lecture 6 I gave only have of the solution

2. Diffraction by cylinder in parabolic approximation

a)



$$\Delta \tilde{u} + k^2 \tilde{u} = 0$$

Let $kR \gg 1$

$$l = R\varphi; \qquad h = r - R$$

$$\tilde{u}(l,h) = e^{ikl}\, u(l,h)$$

Let's write down the Helmholtz equation in new coordinates:

$$\left[\frac{\partial^2}{\partial h^2} + \cancel{\frac{1}{R+h}\frac{\partial}{\partial h}} + \frac{R^2}{(R+h)^2}\frac{\partial^2}{\partial l^2} + k^2\right]\tilde{u}(l,h) = 0$$

Let $h \ll R$

$$\frac{R^2}{(R+h)^2} \approx \frac{1}{(1+\frac{h}{R})^2} \approx 1 - \frac{2h}{R}$$

$$\left[\frac{\partial^2}{\partial h^2} + \frac{\partial^2}{\partial l^2} + k^2 - \frac{2h}{R}\frac{\partial^2}{\partial l^2}\right]\tilde{u}(l,h) = 0$$

$$\tilde{u}(l,h) = e^{ikl}\,u(l,h)$$

$$\Big[\frac{\partial^2}{\partial h^2} + \cancel{\frac{\partial^2}{\partial l^2}} - \cancel{k^2} + \cancel{k^2} + \frac{2h}{R}k^2 - \cancel{\frac{2h}{R}\frac{\partial^2}{\partial l^2}} + 2ik\frac{\partial}{\partial l} - \cancel{\frac{4hik}{R}\frac{\partial}{\partial l}}\Big]u = 0$$

$$k^{-1} \ll H \ll L \ll R$$

characteristic sizes

$$\left(\frac{\partial^2}{\partial h^2} + 2k^2\frac{h}{R} + 2ik\frac{\partial}{\partial l}\right)u(l,h) = 0$$

Introduce dimensionless coordinates:

$$\tau = \frac{l}{2^{\frac{1}{3}}L} = \frac{h}{2^{\frac{1}{3}}R^{\frac{2}{3}}k^{-\frac{1}{3}}}$$

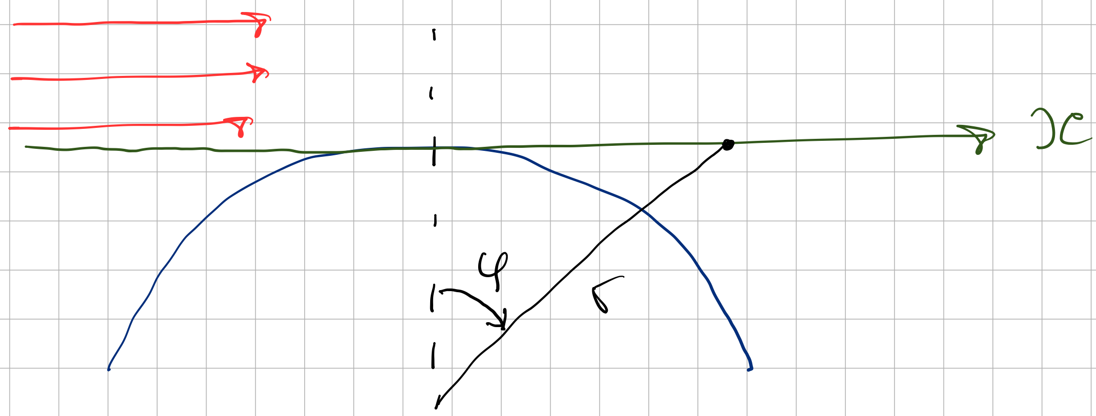
$$\eta = \frac{h}{2^{-\frac{1}{3}}H} = \frac{h}{2^{-\frac{1}{3}}R^{\frac{1}{3}}k^{-\frac{2}{3}}}$$

$$\left(\frac{\partial^2}{\partial \eta^2} + \eta + i\frac{\partial}{\partial \tau}\right)u(\tau,\eta) = 0$$

Equation stays the same for

any pair (k, R) so shoued be soleved once

b) Plane wave



Helmholtz

$$\tilde{u}^{in} = e^{ikx}$$

Parabolic

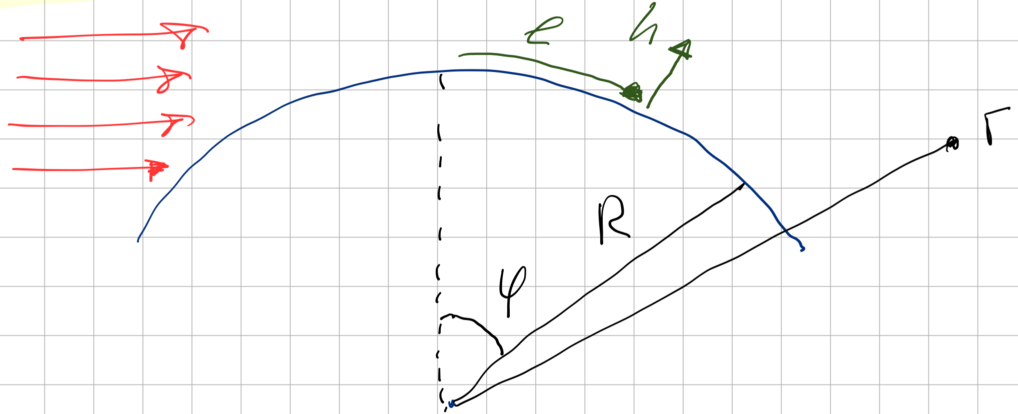
$$u^{in} = e^{ik(x-l)} = e^{ik((R+h)\sin\varphi - R\varphi)}$$

$$= \left| \sin\varphi = \varphi - \frac{\varphi^3}{6} \right| = e^{ik\left(\frac{hl}{R} - \frac{l^3}{6R^2}\right)}$$

in dimensionless coordinates:

$$u^{in} = e^{i\left(\tau\eta - \frac{\tau^3}{3}\right)}$$

c) Problem statement



$u = u^{in} + u^{sc}$

u^{sc} solution to parabolic equation with boundary conditions:

$u^{sc}(\ell, 0) = -u^{in}(\ell, 0) = -e^{-\frac{ik\ell^3}{6R^2}} =$

$= -e^{-\frac{i\tau^3}{3}}$

Solution should satisfy the radiation condition

d) Solution by Fourier transform

$$u^{sc}(\xi, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi\tau}\, u^{sc}(\tau, \eta)\, d\tau$$

$$\left(\frac{\partial^2}{\partial \eta^2} + \eta - \xi\right) u^{sc}(\xi, \eta) = 0$$

Let $z = \xi - \eta$

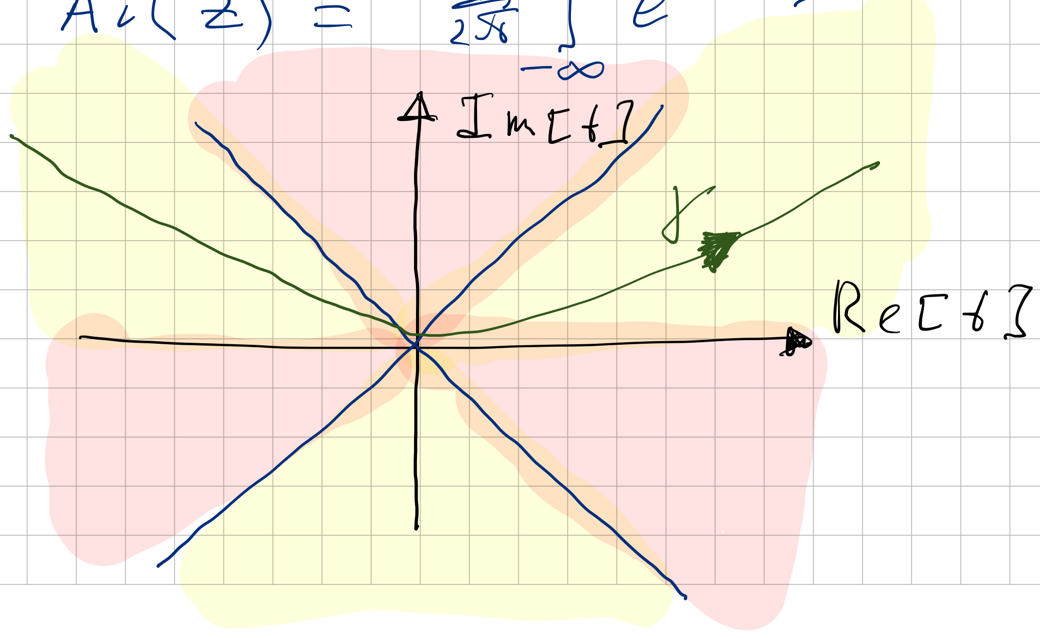
$$\left(\frac{d^2}{dz^2} - z\right) u^{sc}(\xi, z) = 0$$

$\Downarrow$

Airy equation

Solutions:

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3} + tz\right)}\, dt$$



Check that $Ai(z)$ satisfies the equation

$$\frac{d^2}{dz^2}Ai(z) = \frac{1}{2\pi}\int_\gamma e^{it^3 + izt}(-t^2)\,dt =$$

$$= -\frac{1}{2\pi}\int_\gamma (t^2+z)\,e^{\frac{it^3}{3}+itz}\,dt + $$

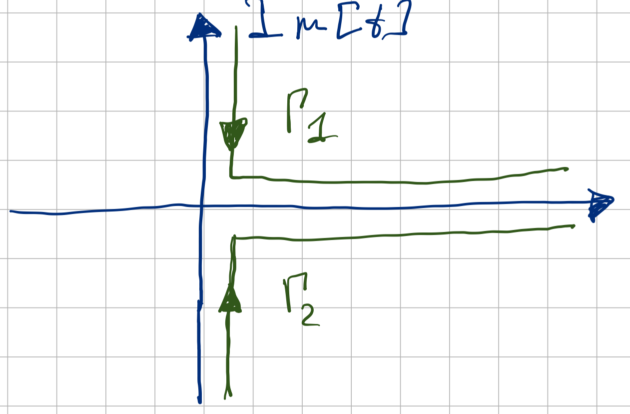
$$+ zAi(z)$$

$$\int_\gamma \frac{d}{dt}\left(e^{\frac{it^3}{3}+itz}\right)dt = 0$$

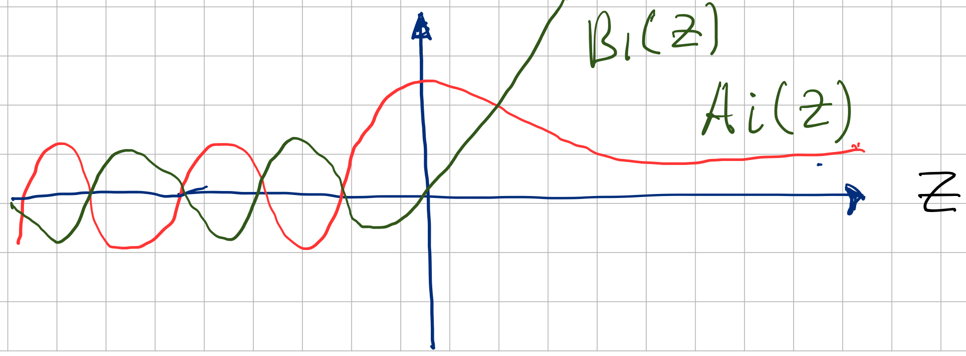
i.e. $\left(\frac{d^2}{dz^2} - z\right)Ai(z) = 0$

Standard choice for second solution is

$$Bi(z) = \frac{1}{2\pi}\int_{\Gamma_1+\Gamma_2} e^{-\frac{t^3}{3}+zt}\,dt$$



For real z:



Asymptotics: $z \gg 1$

$$Ai(-z) \approx \frac{1}{\sqrt{\pi}\, z^{\frac{1}{4}}} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right)$$

$$Bi(z) \approx \frac{1}{\sqrt{\pi}\, z^{\frac{1}{4}}} \cos\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right)$$

Introduce

$$Ci(z) = Bi(z) + i\, Ai(z)$$

$$Di(z) = Bi(z) - i\, Ai(z)$$

$Ci(z)$ — outgoing wave

$Di(z)$ — incoming wave

Finally,

due to radiation condition

$$U^{sc}(\xi, \eta) = A(\xi)\, Ci(\xi - \eta) + \cancel{B(\xi)\, Di(\xi - \eta)}$$

Find $A(\xi)$ from boundary conditions.

$$\int_{-\infty}^{\infty} e^{i\xi\tau} A(\xi)\, Ci(\xi)\, d\xi = -e^{-\frac{i\tau^3}{3}}$$

Inverse Fourier gives.

$$A(\xi)\, Ci(\xi) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{i\tau^3}{3} - i\tau\xi} d\tau = -Ai(\xi)$$

$$A(\xi) = -\frac{Ai(\xi)}{Ci(\xi)};$$

V. A. Fock result

Solution:

$$U^{sc}(\tau, \eta) = -\int_{-\infty}^{\infty} e^{i\xi\tau} \frac{Ai(\xi)}{Ci(\xi)}\, Ci(\xi - \eta)\, d\xi.$$