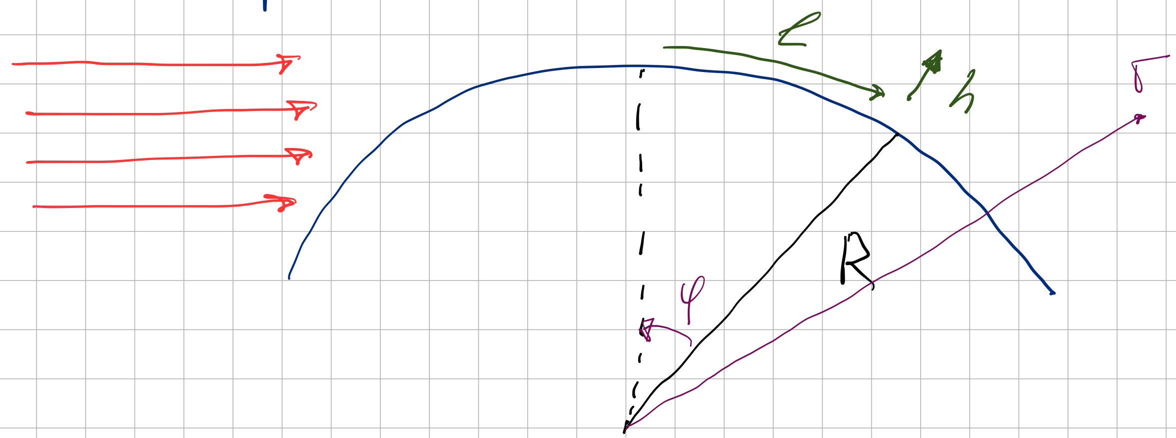
Lecture 8

Asymptotic study of V. A Fock solution

Recap



$\ell = R\varphi \qquad h = r - R$

Dimensionless variables

$$\tau = \frac{\ell}{2^{\frac{1}{3}} R^{\frac{2}{3}} k^{-\frac{1}{3}}}$$

$$\eta = \frac{h}{2^{-\frac{1}{3}} R^{\frac{1}{3}} k^{-\frac{2}{3}}}$$

$$\left(\frac{\partial^2}{\partial \eta^2} + \eta + i\frac{\partial}{\partial \tau}\right) U(\tau, \eta) = 0$$

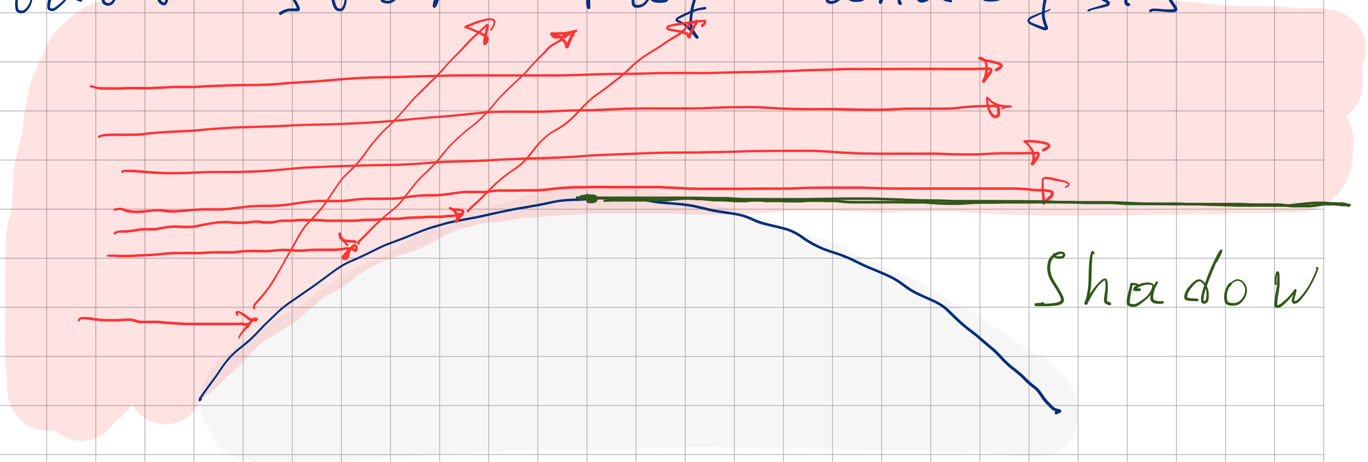
$$U = u^{in} + u^{sc}$$

$$u^{in} = e^{i\left(\tau\eta - \frac{\tau^3}{3}\right)}$$

$$u^{sc}(\tau, \eta) = -\int_{-\infty}^{\infty} e^{i\zeta\tau} \frac{Ai(\zeta)}{Ci(\zeta)} Ci(\zeta - \eta)\, d\zeta$$

Let's try to interpret this formula

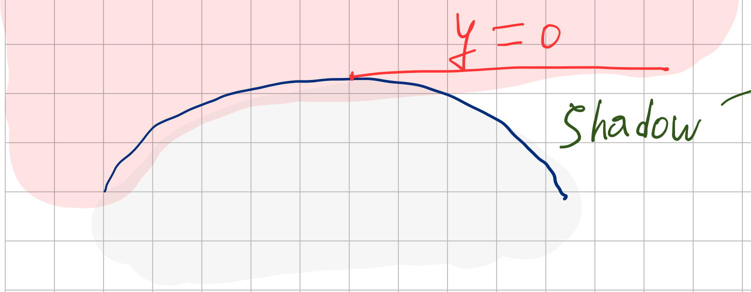
a) Start from Ray analysis

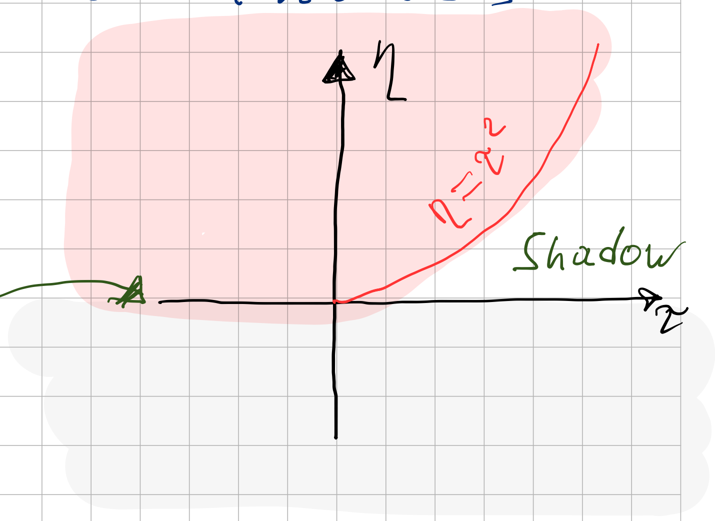


Critical ray $y=0$

In deformed coordinates

$$\eta = \tau^2$$





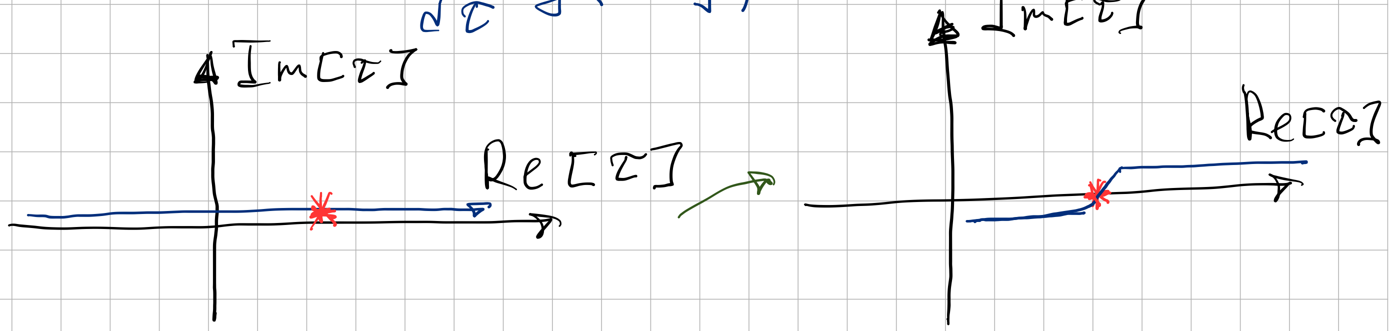
Homework: Implement V. A. Fock formula in Matlab (~10 lines), check that shadow starts around $\eta = \tau^2$

b) Analysis using Fake saddle point method

$$u(\tau) = \int_{-\infty}^{\infty} f(\tau)\, e^{ig(\tau)}\, d\tau$$

Look for points τ_{*j} on real axis such that

$$\frac{d}{d\tau} g(\tau_{*j}) = 0$$



$$u = \sum_j f(\tau_{*j})\, e^{i g(\tau_{*j}) \pm i\frac{\pi}{4}} \sqrt{\frac{2\pi}{|g''(\tau_{*j})|}}$$

$+$ for positive g''

$-$ for negative g''

b') Let's demonstrate on plane wave

$$u^{in} = \exp\left\{i\left(\tau\eta - \frac{\tau^3}{3}\right)\right\}$$

$$\mathcal{F}[u^{in}] = \int \exp\left\{i\left(\tau(\eta-\xi) - \frac{\tau^3}{3}\right)\right\} d\xi =$$

$$= Ai(\xi-\eta)\ ; \text{ i.e.}$$

$$u^{in} = \int_{-\infty}^{\infty} e^{i\tau\xi} Ai(\xi-\eta)\, d\xi$$

Estimate the latter

Ignore part where $\xi-\eta > 0$,

since $Ai(\xi-\eta)$ decays there.

and use Asymptotics for $\xi-\eta<0$

$$Ai(-z) = \frac{1}{\sqrt{\pi}\, z^{\frac{1}{4}}} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right)$$

$$u^{in} \approx \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\eta} \frac{1}{(\eta-\xi)^{\frac{1}{4}}} e^{i\xi\tau} \cdot$$

$$\Big(\underbrace{\exp\Big\{ i\,\frac{2}{3}(\eta-\xi)^{3/2} - \frac{i\pi}{4} \Big\}}_{} + \underbrace{\exp\Big\{ -i\,\frac{2}{3}(\eta-\xi)^{3/2} + \frac{i\pi}{4} \Big\}\Big) \, d\xi}_{}$$

Study the first term

$$g(\xi) = \frac{2}{3}(\eta-\xi)^{3/2} + \xi\tau$$

$$g'(\xi) = -\sqrt{\eta-\xi} + \tau$$

has saddle points for $\tau > 0$

$$\xi_* = \eta - \tau^2$$

Using saddle point formula:

$$u_1^{in} \approx \exp\Big\{ i\eta\tau - i\frac{\tau^3}{3} \Big\}, \quad \tau > 0$$

Second term

$$u_2^{in} \approx \exp\Big\{ i\eta\tau - i\frac{\tau^3}{3} \Big\}, \quad \tau < 0$$

Thus, Fake saddle point method provides the correct estimate

c) Fock solution in shadow zone

$$u^{sc}(\tau,\eta) = -\int_{-\infty}^{\infty} e^{i\xi\tau} \frac{Ai(\xi)}{Ci(\xi)} Ci(\xi-\eta)\, d\xi$$

Let's show that for $\eta < \tau^2$ u^{sc} cancels u^{in} asymptotically

For $z > 0$ $Ci(z)$ grows as $Bi(z)$

$$Ci(z) \simeq \frac{1}{\sqrt{\pi}\, z^{\frac{1}{4}}} \exp\left\{\frac{2}{3} z^{3/2}\right\},$$

$$Ci(-z) \simeq \frac{1}{\sqrt{\pi}\, z^{\frac{1}{4}}} \exp\left\{i\frac{2}{3} z^{3/2} + \frac{i\pi}{4}\right\}$$

Thus $\dfrac{Ci(\xi-\eta)}{Ci(\xi)}$ small for positive ξ and η

Thus $\int_{-\infty}^{\infty} \simeq \int_{-\infty}^{0}$

Using asymptotics for Ai and Ci we have

$$u^{sc} \simeq u_1^{sc} + u_2^{sc}$$

$$u_1^{sc} = -\frac{e^{i\pi/4}}{2\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{(\eta-\xi)^{\frac{1}{4}}} \exp\left\{i\xi\tau + i\frac{2}{3}(\eta-\xi)^{3/2}\right\} d\xi$$

$$u_2^{sc} = -\frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{(\eta-\xi)^{\frac{1}{4}}} \exp\left\{i\xi\tau - i\frac{4}{3}(-\xi)^{3/2} + i\frac{2}{3}(\eta-\xi)^{3/2}\right\} d\xi$$

Similarly to previous one

$u_1^{sc} \simeq -u^{in}$; $\tau > 0$

u_2^{sc} does not have a saddle point

Introduce $s = -\xi$; $s > 0$

$$g(s) = -s\tau - \frac{4}{3} s^{3/2} + \frac{2}{3}(\eta + s)^{3/2}$$

$$g'(\xi) = -\tau - 2\xi^{\frac{1}{2}} + (\xi+\eta)^{\frac{1}{2}}$$

$g'(\xi)$ has no positive roots

Indeed

$$(\xi+\eta)^{\frac{1}{2}} = 2\xi^{\frac{1}{2}} + \tau$$

$$< \xi^{\frac{1}{2}} + \eta^{\frac{1}{2}} \;; \quad \xi, \eta > 0$$

In shadow $\tau > \eta^{\frac{1}{2}}$, i.e.

$$(\xi+\eta)^{\frac{1}{2}} < \xi^{\frac{1}{2}} + \eta^{\frac{1}{2}} < 2\xi^{\frac{1}{2}} + \tau$$

If we consider illuminated zone then u_2^{sc} will provide reflected wave.

Remark Additional properties of Airy functions

Airy equation has rotation symmetry

$$\left(\frac{d^2}{dz^2} - z\right) V(z) = 0$$

If $V(z)$ is a solution, then $V(z e^{\pm\frac{2\pi i}{3}})$ also a solution

$$C_i(z) = 2e^{\frac{i\pi}{6}} Ai(e^{\frac{2\pi i}{3}} z)$$

*One can proove through Asymptotic analysis.

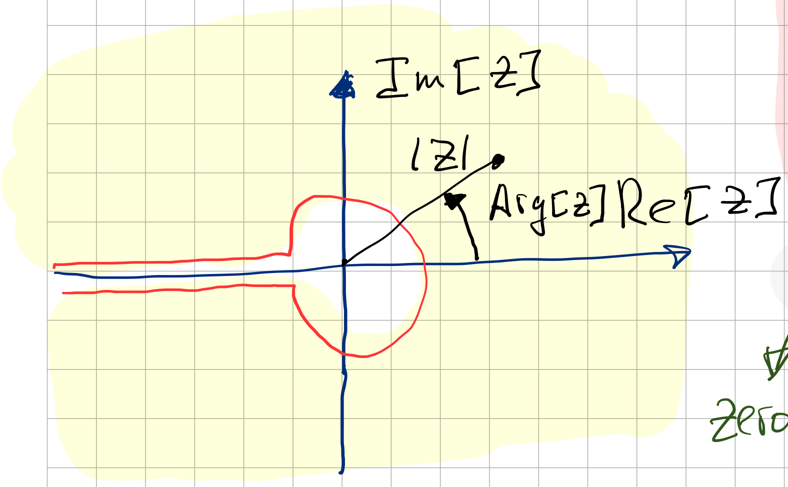
Now let us find zeros of $Ai(z)$

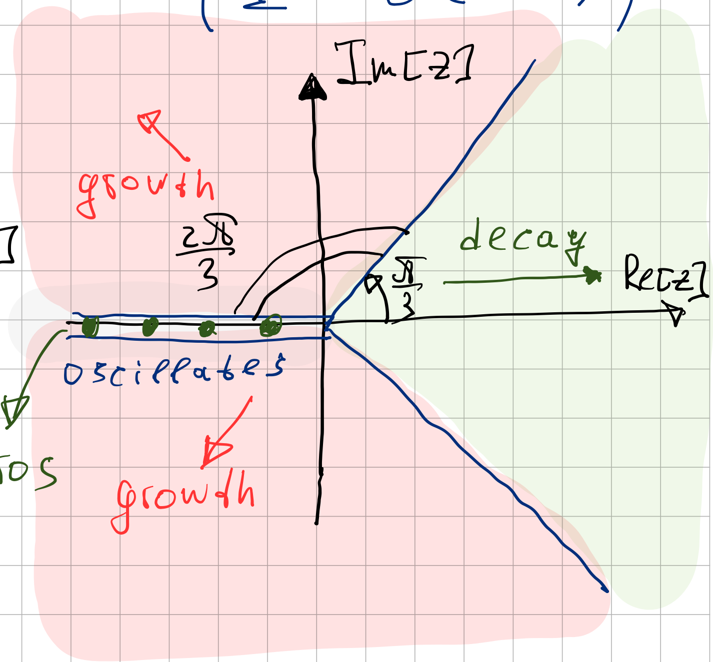
Airy equation has one singular point (irregular) at ∞, solutions do not branch

General theory says that for

$|z| \gg 1$ $\quad -\pi < \text{Arg}[z] < \pi$

$$Ai(z) = \frac{1}{2\sqrt{\pi}\, z^{\frac{1}{4}}}\, e^{-\frac{2}{3} z^{3/2}} \left(1 + O\left(z^{-3/2}\right)\right)$$



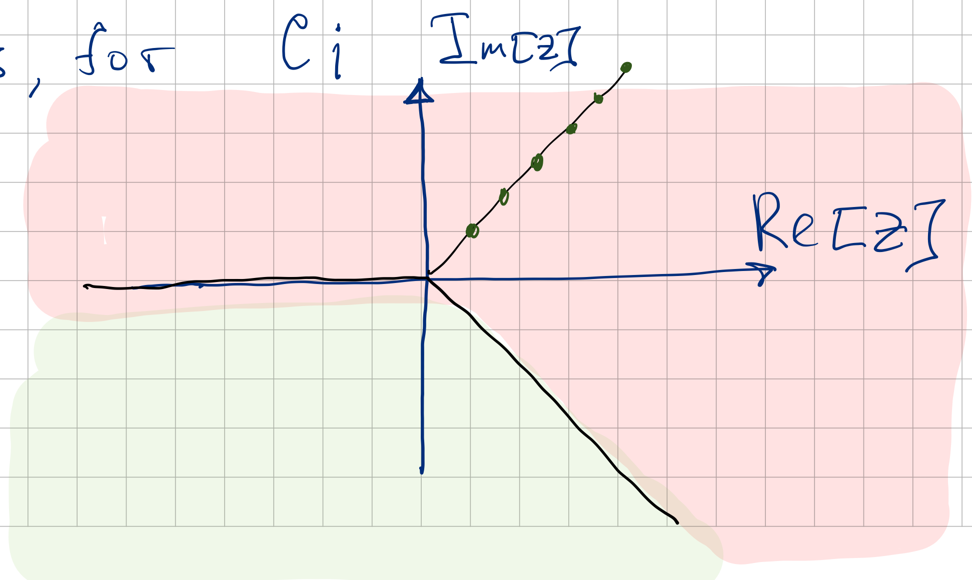


At $\text{Arg}[z] = \pi$ the Asymptotic is given by a sum for $\text{Arg}[z] = -\pi$ and $\text{Arg}\, z = \pi$.

So, zeros lie on negative real axis

$$z_n \approx -\left(\frac{3}{2}\left(\pi n - \frac{\pi}{4}\right)\right)^{2/3}$$

Thus, for Ci

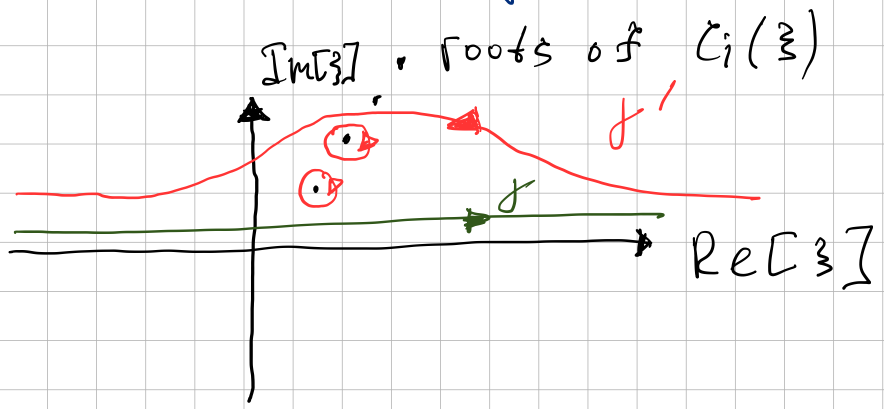


For details see F. Olver

Asymptotics and special function

d) Creeping waves

Zeros of $Ci(\xi)$ are poles in V. A. Fock integral.



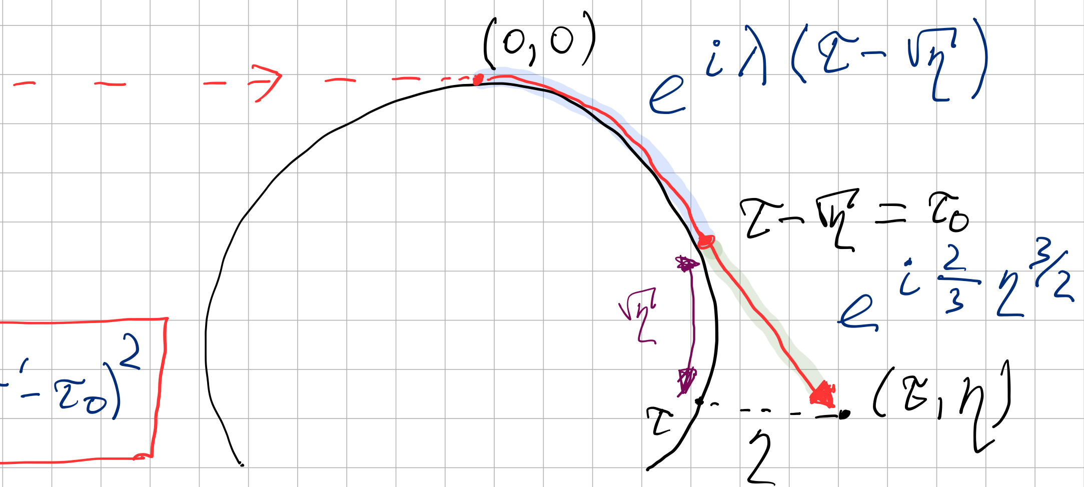
Denote λ as some root of $Ci(\xi)$

Pole contribution is

$$\tilde{u}^{sc}_{\lambda} = -2\pi i \frac{Ai(\lambda)}{Ci'(\lambda)} e^{i\tau\lambda} Ci(\lambda-\eta)$$

for $\eta \gg \lambda$

$$e^{i\tau\lambda} Ci(\lambda-\eta) \approx \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}\,\eta^{\frac{1}{4}}} \exp\left\{ i\frac{2}{3}\eta^{3/2} + i\lambda(\tau-\sqrt{\eta}) \right\}$$



$$\boxed{\eta' = (\tau' - \tau_0)^2}$$

equation of tangent line

$$u^{sc} \approx \tilde{u}^{sc} + u^{sc}_{cr}$$

$$u^{sc}_{cr}(\tau, \eta) = \sum_j a_j e^{i\lambda_j \tau} \operatorname{Ci}(\lambda_j - \eta)$$