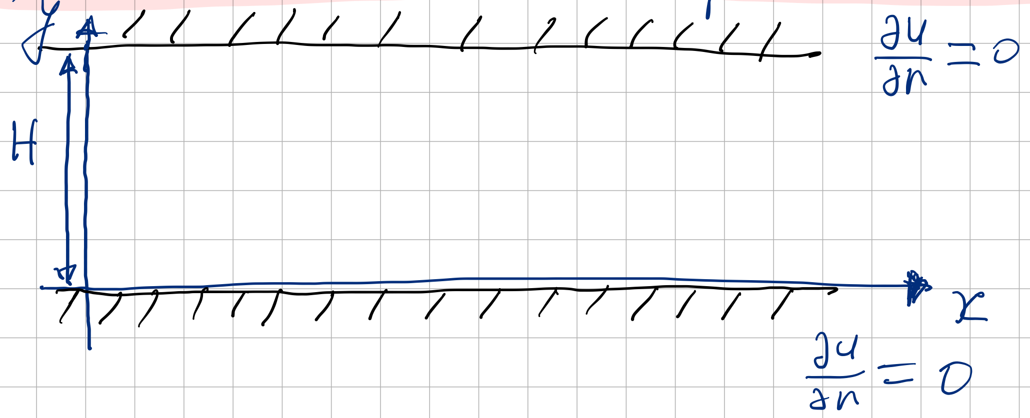
# Lecture 9. Waveguides

## 1) Modes of a plane waveguide




$$\Delta u + k^2 u = 0$$

$$u = e(x)\, p(y)$$

$$\frac{d^2 e}{dx^2} + k_x^2 e = 0 \qquad\qquad \frac{d^2 p}{dy^2} + k_y^2 p = 0$$

$$k_x^2 + k_y^2 = k^2 = \frac{\omega^2}{c^2}$$

a) $e = e^{ik_x x} + e^{-ik_x x}$

b) Sturm - Liouville (SL) problem

$$\frac{d^2 p}{dy^2} + k_y^2 p = 0 \qquad \frac{dp}{dy}(0) = \frac{dp}{dy}(H) = 0$$

Find $k_y$ for which solution exist

$$p(y) = A\cos(k_y y) \qquad k_y = \frac{\pi n}{H}$$

$$k_x^n = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi n}{H}\right)^2}$$

Thus

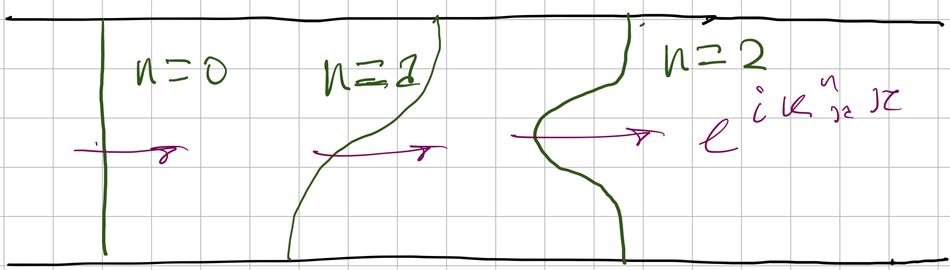
$$U_f = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n}{H} y\right) e^{i k_x^n x} =$$

$$= \sum_{n=0}^{\infty} \tilde{A}_n \left[ e^{i k_x^n x - i\left(\frac{\pi n}{H}\right) y} + \right.$$

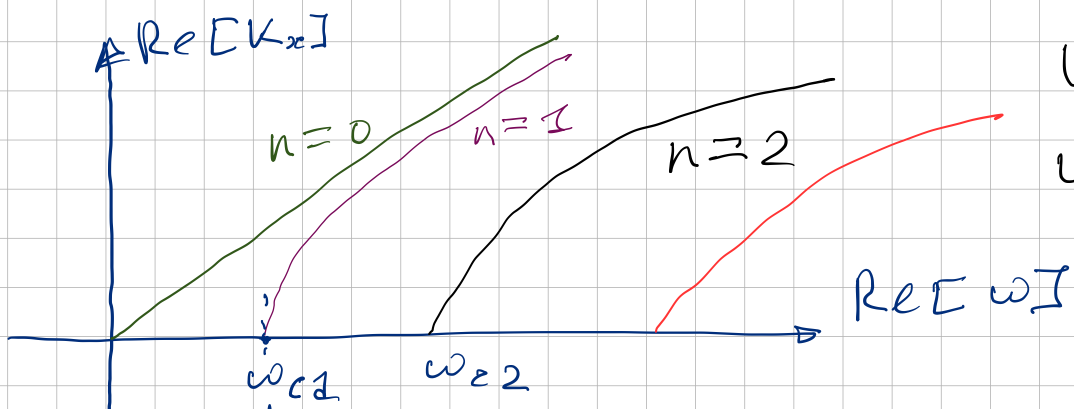
$$\left. + e^{i k_x^n x + i\left(\frac{\pi n}{H}\right) y} \right]$$

Brillouin (partial) wave

Modes:




Partial waves




$$V_{ph} = \frac{\omega}{k_x}$$

$$V_{gr} = \frac{d\omega}{dk_x}$$

cut-off frequency

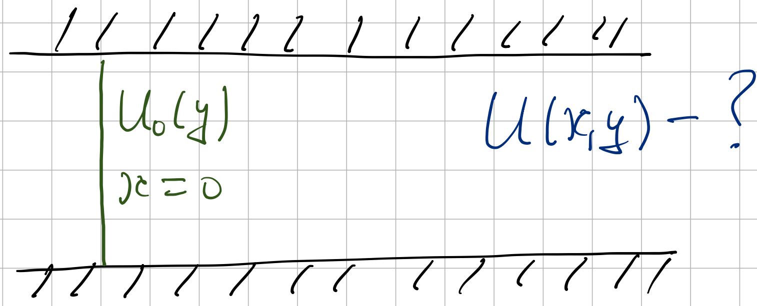
Note that modes are ortogonal:

$$\int_0^H \cos\left(\frac{\pi m}{H} y\right) \cos\left(\frac{\pi n}{H} y\right) = 0 .$$

if $m \neq n$

Any wavefield can be presented in terms of mode (Fourier) series

Indeed, consider an excitation problem



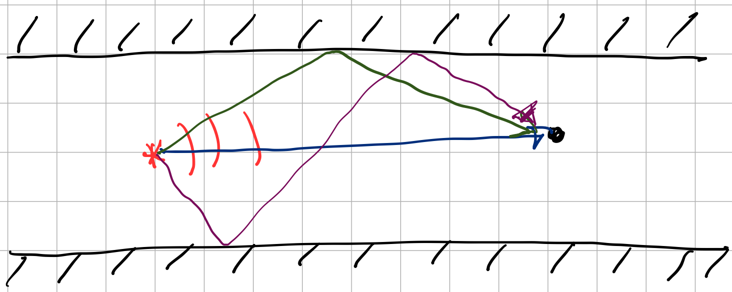

$$U_0(y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n}{H} y\right)$$

$$A_0 = \frac{1}{h} \int_0^h U_0(y)\, dy$$

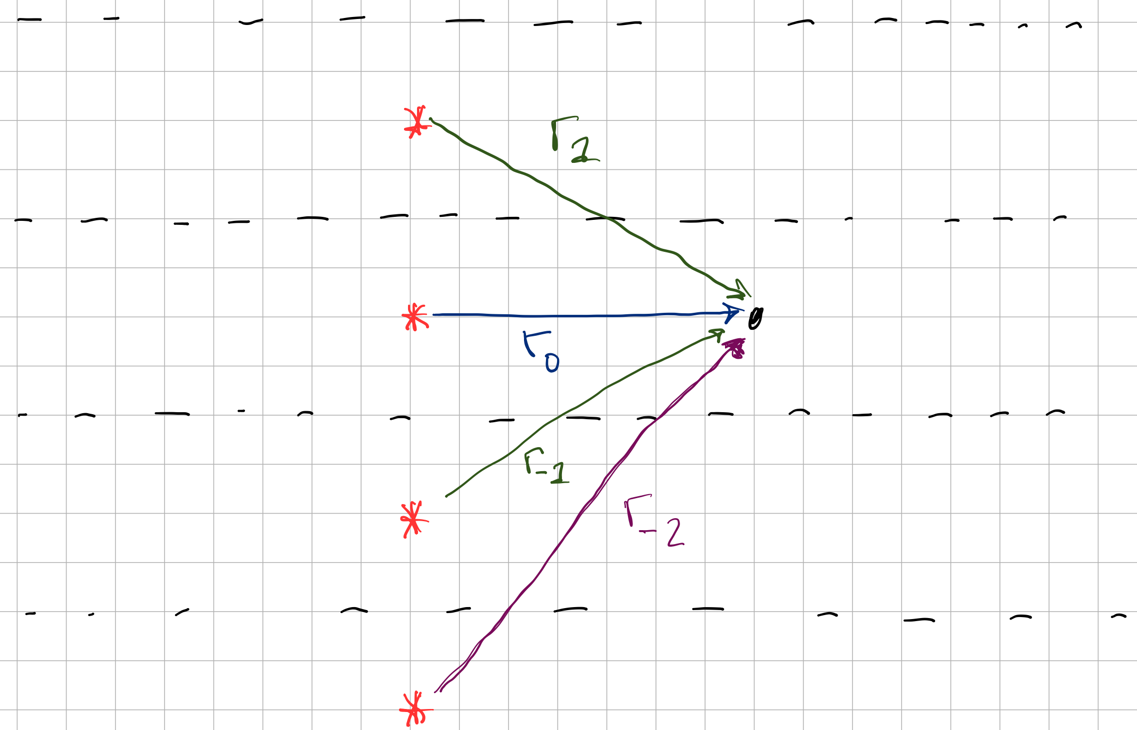
$$A_n = \frac{2}{h} \int_0^h U_0(y) \cos\left(\frac{\pi n}{H} y\right) dy \,; \; n>0$$

$$U(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\pi n}{H} y\right) e^{i k_x^n x}$$

c) Green's function of a waveguide

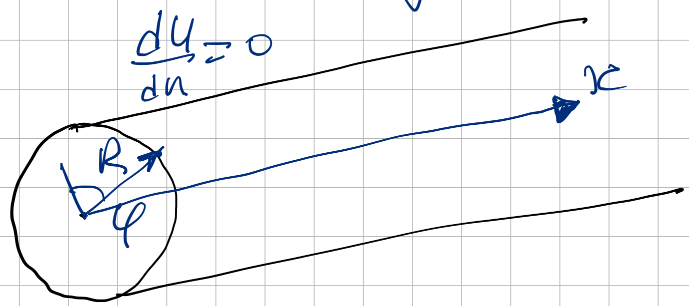


One approach is to use mode decomposition as above.
Another one is to apply method of reflections.




$$G(x,y) = -\frac{i}{4}\sum_{n=-\infty}^{\infty} H_0^2(k r_n)$$

## 2) Circular waveguide




$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0$$

$$u = \sum_{m,n=0}^{\infty} A_{m,n}\, e^{i\varphi m + k_{m,n} x}\, F_{m,n}(r)$$

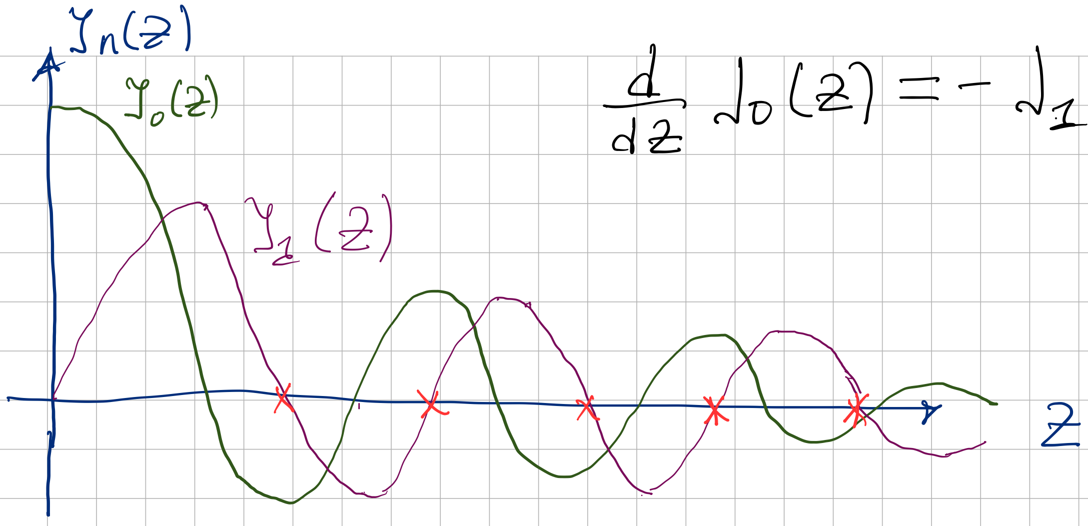
$$\frac{d^2 F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \left(k^2 - k^2_{m,n} - \frac{m^2}{r^2}\right) F_{m,n} = 0$$

$$F_{m,n} = J_m\left(\sqrt{k^2 - k^2_{m,n}}\, r\right)$$

$$\frac{dF_{m,n}}{dr} = 0 \Rightarrow \frac{dJ_m}{dr}\left(\sqrt{k^2 - k^2_{m,n}}\, r\right) = 0$$

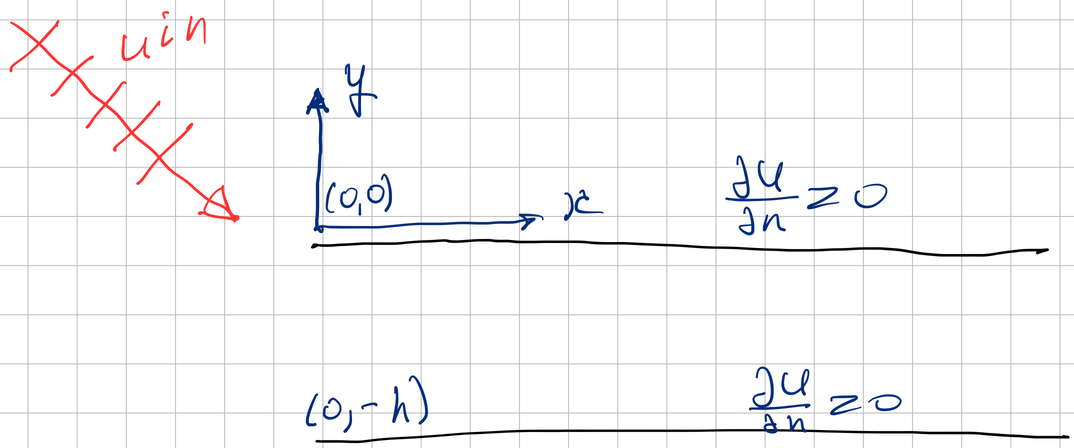
\# modes are zeros of derivative

$$\frac{d}{dz} J_0(z) = -J_1(z)$$




## 3) Diffraction by an open-ended plane waveguide. Weinstein's problem

### a) Statement




$$u = u^{sc} + u^{in}$$

$$\Delta u + k^2 u = 0$$

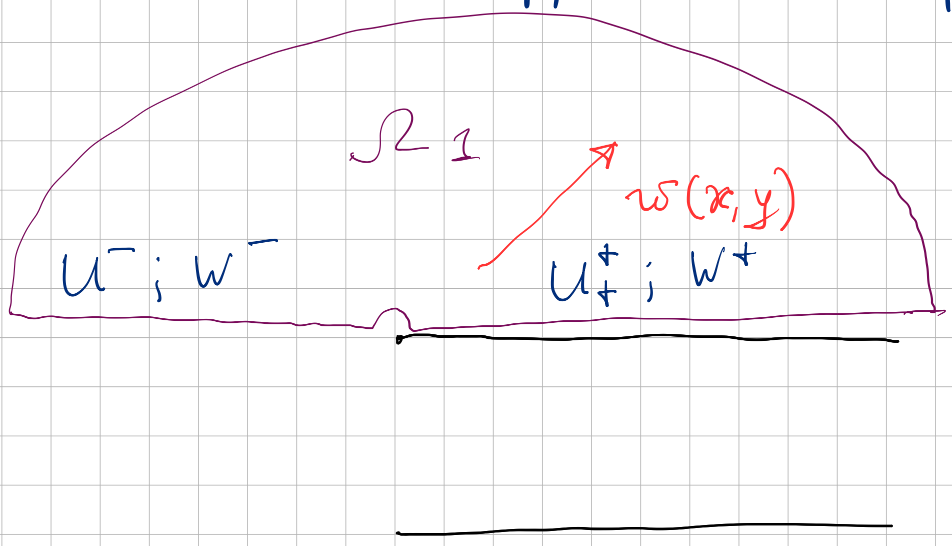
Meixner and radiation conditions

### б) Derivation of Wiener-Hopf equation using Green's identity

Auxiliary function:

$$w(x,y) = e^{i\xi x + i\sqrt{k^2 - \xi^2}\, y}$$

# Domain 1: Upper Half-plane




$$\int_{\partial\Omega_1}\left[\frac{\partial w}{\partial n}u^{sc}-\frac{\partial u^{sc}}{\partial n}w\right]dl=0$$

$$-i\sqrt{k^2-\xi^2}\,U^- + W^- - i\sqrt{k^2-\xi^2}\,U_+^+ + \underbrace{W^+}_{\text{Forcing}} = 0$$

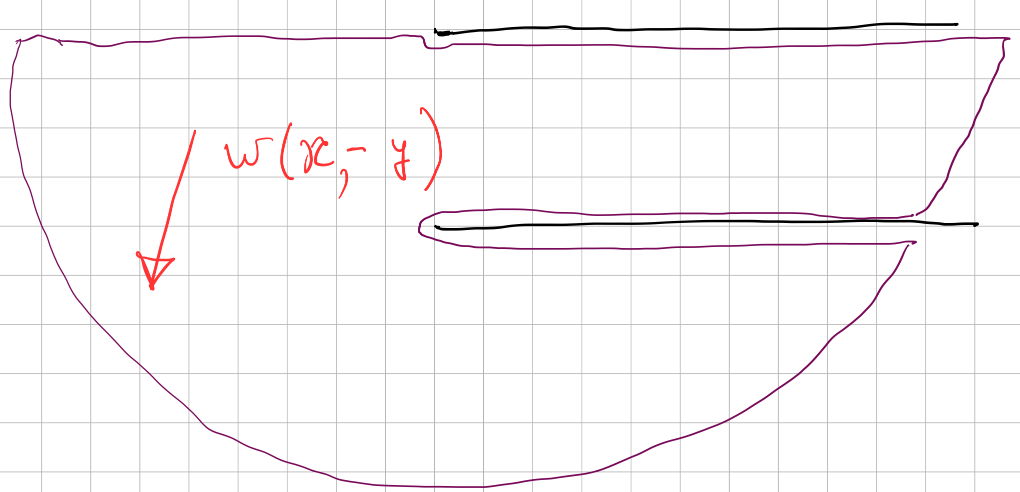
$$U^- = \int_{-\infty}^{0} e^{i\xi x}\,u^{sc}(x,0)\,dx$$

$$U_+^+ = \int_{0}^{\infty} e^{i\xi x}\,u^{sc}(x,0^+)\,dx$$

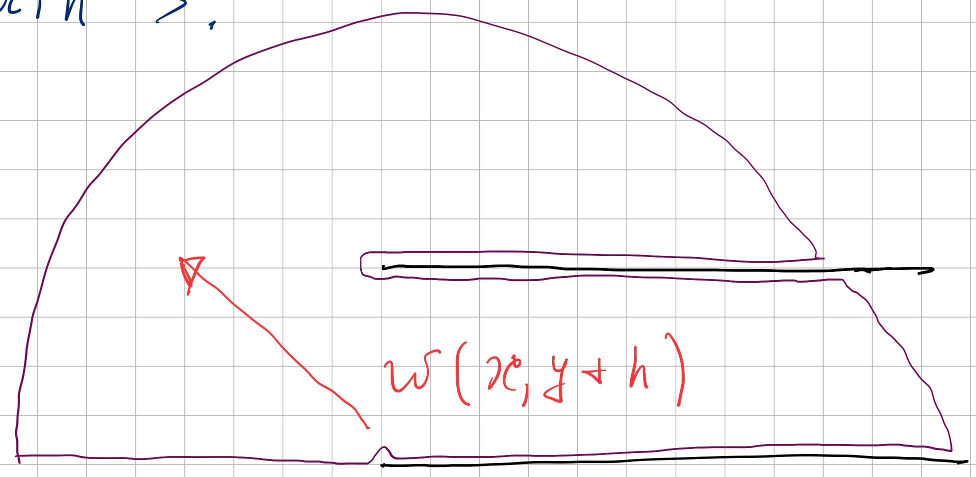
$$W^- = \int_{-\infty}^{0} e^{i\xi x}\,\frac{\partial u^{sc}}{\partial y}(x,0)\,dx$$

$$W^+ = -\int_{0}^{\infty} e^{i\xi x}\,\frac{\partial u^{in}}{\partial y}(x,0)\,dx$$

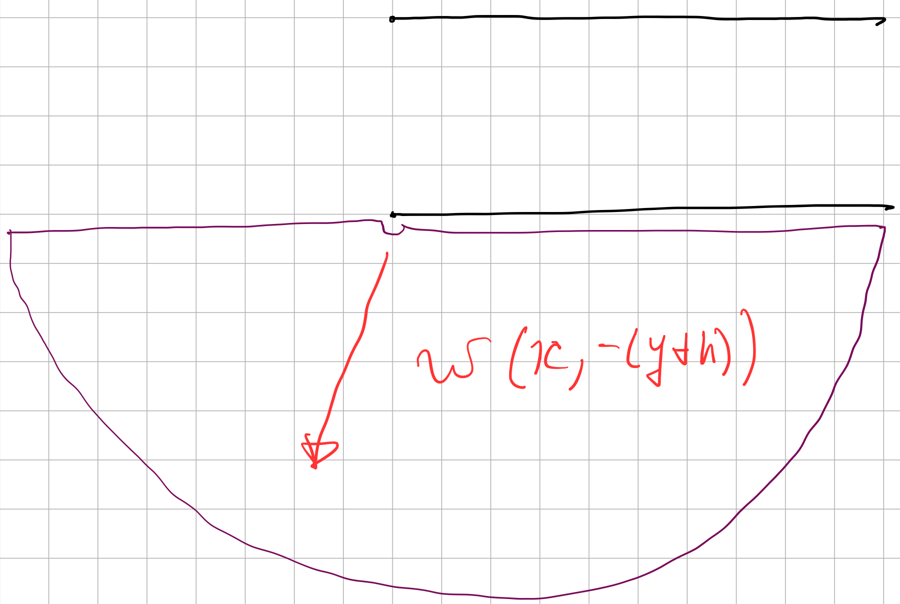
# Domain 2:



Domain 3:




Domain 4:




We got 3 more functional equations, which are reduced to 2 by 2 matrix W-H:

$$\Psi^- = K\Psi^+ + F$$

$$\Psi^- = \begin{pmatrix} W^- \\ \Xi^- \end{pmatrix}; \quad \Psi^+ = \begin{pmatrix} U^+ \\ \Phi^+ \end{pmatrix}$$

$$\Phi^+ = \int_0^\infty e^{i\xi x}\left\{u^{sc}(x,-h^+) - u^{sc}(x,-h^-)\right\}dx$$

$$\Xi^- = \int_{-\infty}^{0} e^{i\xi x} \frac{\partial u^{sc}}{\partial y}(x,-h)\, dx$$

$$U^+ = U^+_+ - U^+_-$$

$$K = \frac{i\gamma}{2}\begin{pmatrix} 1 & e^{i\gamma h} \\ e^{i\gamma h} & 1 \end{pmatrix}; \quad \gamma = \sqrt{k^2-\xi^2}$$

$$F = \frac{k\sin\theta^{in}}{\xi - k\cos\theta^{in}}\begin{pmatrix} 1 \\ e^{ikh\sin\theta^{in}} \end{pmatrix}$$

System can be decoupled. Indeed

$$K = -\frac{i\gamma}{4}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1-e^{i\gamma h} & 0 \\ 0 & 1+e^{i\gamma h} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Introduce

$$\tilde{\psi}^- = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\psi^- \quad \tilde{\psi}^+ = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\psi^+$$

$$\tilde{F} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} F$$

$$\tilde{\psi}^- = \frac{i\gamma}{2}\begin{pmatrix} 1-e^{i\gamma h} & 0 \\ 0 & 1+e^{i\gamma h} \end{pmatrix}\tilde{\psi}^+ + \tilde{F}$$

2 independent scalar equations that are solved using scalar W-H method

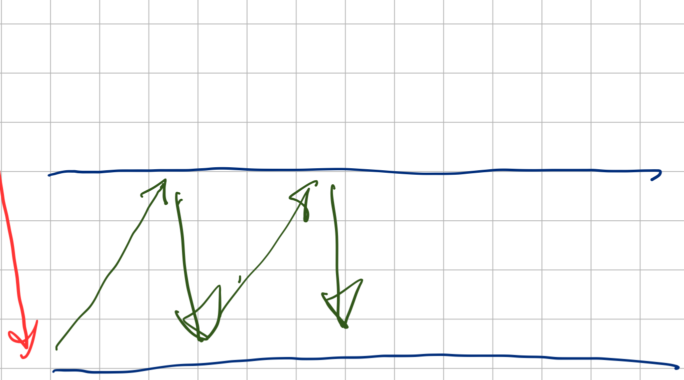
Homework: factorize

$$f(\xi) = f_+(\xi)\, f_-(\xi)$$

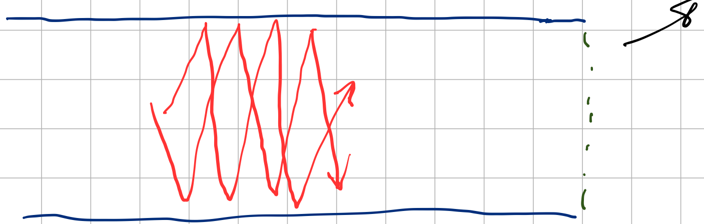
$$f(\xi) = \sqrt{k^2-\xi^2}\left(1 \pm e^{i\sqrt{k^2-\xi^2}\, h}\right)$$

# Speculations

a) Grazing wave excites waveguide



b) Vise-versa Mode close to cut-off reflects almost completely from the open end:



Reflects from open end with $R \simeq -1$.

mode close to cut-off

Home work***: Formulate W-H problem for waveguide mode, show that almost complete reflection happens close to cut-off

Such modes form standing waves in Fabry-Perot resonators: