The Wiener–Hopf perspective on the embedding formula: new results

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Outline

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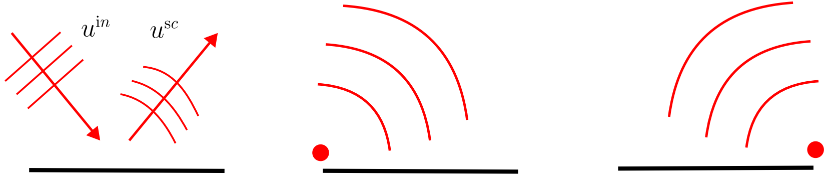
Standard approach to embedding

A standard way to derive the embedding formula is as follows[1]:

- Find operator, H so that applied to the total field u, the result $H[u]$ is still a solution to the Helmholtz equation with the same boundary conditions (excluding the tip conditions) and $H[u^{\mathrm{in}}] = 0$.
- Define the edge Green's functions v_i by placing line source(s) near the tips so that they have the same behaviour as $H[u]$ at the tips.
- Show that $L(u) = H[u] + \sum_i K_i v_i$ satisfies the tip conditions, with some constants K_i that are to be determined. Since $L(u)$ is the solution of the diffraction problem, then, by uniqueness $L(u) = 0$, giving the *weak form of embedding*.
- Take the far-field limit and use reciprocity to obtain the *strong embedding formula*.

[1] Craster, Shanin, and Doubravsky, "Embedding formulae in diffraction theory".

Example: strip problem



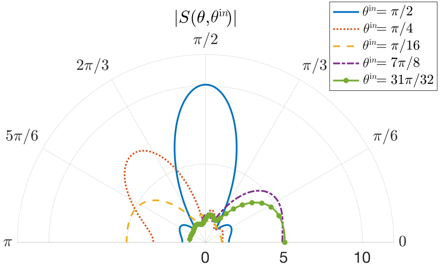
$S(\theta, \theta^{\mathrm{in}})$ $S_1(\theta)$ $S_2(\theta)$

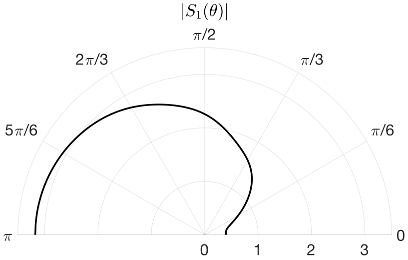
$$H = \frac{\partial}{\partial x} + ik\cos\theta^{\mathrm{in}}$$

$$S(\theta, \theta^{\mathrm{in}}) = \frac{S_2(\theta^{\mathrm{in}})S_2(\theta) - S_1(\theta^{\mathrm{in}})S_1(\theta)}{k\cos\theta + k\cos\theta^{\mathrm{in}}}$$

Computational benefit: Unknowns depend only on one variable

Strip: numerics





Knowing just $S_1(\theta)$ and $S_2(\theta) = S_1(\pi - \theta)$ we can $S(\theta, \theta^{\text{in}})$ for any θ^{in}.

Wiener–Hopf perspective

$$\mathrm{U}^-(t, z_i) = \mathrm{K}(t)\mathrm{U}^+(t, z_i) + \frac{\mathrm{r}}{z - z_i}, \quad t \in \mathbb{R} \quad (1)$$

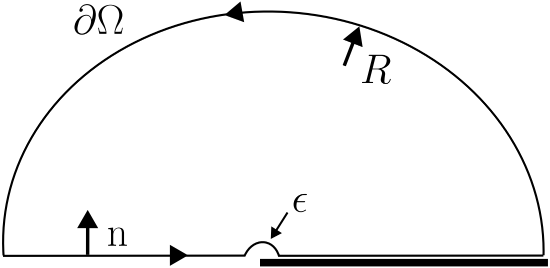
For polar forcing embedding follows from the method of normal solutions[2]:

- Find N linearly independent solutions of (1) without a forcing term
- Compose a matrix $\mathrm{X}^{\pm}$ of solutions. $\mathrm{X}^{\pm}$ is called the matrix of normal solutions if $\det|\mathrm{X}^{\pm}| = 0$.
- Factorise the kernel $\mathrm{K}(z) = \mathrm{X}^-(z)(\mathrm{X}^+(z))^{-1}$
- Apply Liouville's theorem to get the solution:

$$\mathrm{U}^+(z, z_i) = -\frac{\mathrm{X}^+(z)(\mathrm{X}^-(z_i))^{-1}\mathrm{r}}{z - z_i}$$

[2] Gakhov, "Riemann's boundary problem for a system of n pairs of functions".

Green's formula as a Wiener–Hopf tool



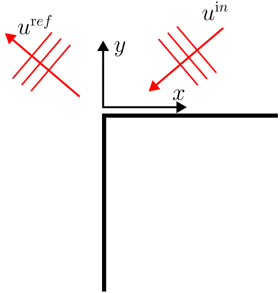
$$\int_{\partial\Omega}\left[w\frac{\partial u^{\mathrm{sc}}}{\partial n} - u^{\mathrm{sc}}\frac{\partial w}{\partial n}\right] dl = 0$$

$$w = \exp\{izx + i\gamma(z)y\}, \quad \gamma(z) = \sqrt{k^2 - z^2}$$

The integral on Ω tends to zero due to radiation conditions, and the rest gives us a functional relation between spectral functions

The wedge problem

$$(\Delta + k^2)u(x, y) = 0$$
$$u = u^{\mathrm{in}} + u^{\mathrm{ref}} + u^{\mathrm{sc}}$$
$$u|_{\mathrm{wedge}} = 0$$



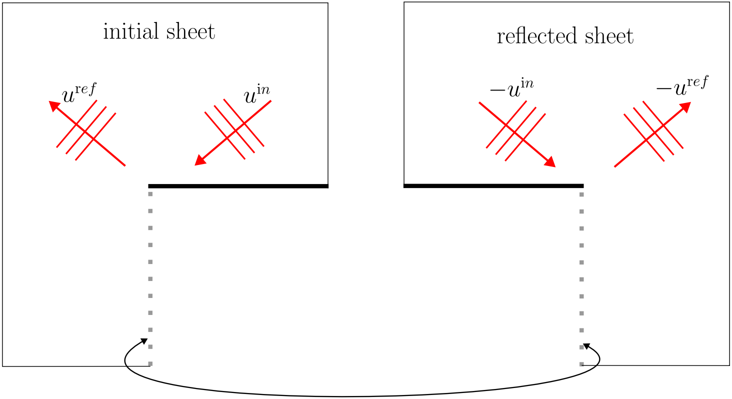
Near-field behaviour:

$$u^{\mathrm{sc}} = Cr^{2/3}\sin(2/3\theta) + O(r^{4/3})$$

Far-field behaviour:

$$u^{\mathrm{sc}}(r, \theta) = -\frac{\exp\{ikr - i\pi/4\}}{\sqrt{2\pi kr}} S(\theta, \theta^{\mathrm{in}}) + O\left(\frac{1}{r}\right).$$

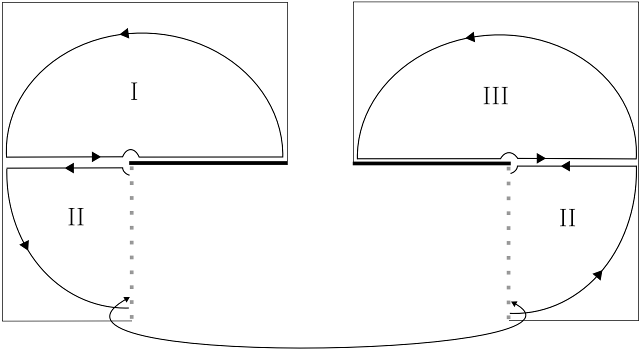
Reformulation on a branched manifold with a boundary



Apply Green's theorem 3 times with

$$w_1 = \exp\{izx + i\gamma(z)y\}$$
$$w_2 = \exp\{izx - i\gamma(z)y\}$$

Reformulation on a branched manifold with a boundary



$$Y_i^- = \int_{-\infty}^{0} \exp\{izx\} u_i^{\mathrm{sc}}(0,x)dx, \quad W_i^- = \int_{-\infty}^{0} \exp\{izx\} \frac{\partial u_i^{\mathrm{sc}}}{\partial y}(-0,x)dx,$$

$$Y_i^+ = \int_{0}^{\infty} \exp\{izx\} u_i^{\mathrm{sc}}(0,x)dx, \quad W_i^+ = \int_{0}^{\infty} \exp\{izx\} \frac{\partial u_i^{\mathrm{sc}}}{\partial y}(-0,x)dx.$$

Wiener-Hopf equation

$$\mathrm{U}^-(z, z_i) = \mathrm{K}(z)\mathrm{U}^+(z, z_i) + \mathrm{F}(z, z_i)$$

$$\mathrm{U}^-(z, z_i) = \begin{pmatrix} W_2^-(z, z_i) \\ W_1^-(z, z_i) \\ Y_1^-(z, z_i) \end{pmatrix}, \quad \mathrm{U}^+(z, z_i) = \begin{pmatrix} W_1^+(z, z_i) \\ W_2^+(z, z_i) \\ Y_2^+(z, z_i) \end{pmatrix}$$

$$\mathrm{K} = \frac{i}{2\gamma} \begin{pmatrix} 0 & 2i\gamma & 2\gamma^2 \\ i\gamma & i\gamma & -\gamma^2 \\ -1 & 1 & i\gamma \end{pmatrix}, \quad \mathrm{F} = -\begin{pmatrix} \dfrac{2k \sin \theta^{\mathrm{in}}}{z + z_i} \\ \dfrac{k \sin \theta^{\mathrm{in}}}{(z - z_i)} \\ \dfrac{ik \sin \theta^{\mathrm{in}}}{(z - z_i)\gamma} \end{pmatrix}.$$

Aitken, "On the factorisation of matrix Wiener–Hopf kernels arising from acoustic scattering problems"

Reducing the equation

$$
\mathrm{K} \to \mathrm{PKP}^{-1} = \frac{1}{2\gamma}\begin{pmatrix} -1 & 0 & 0 \\ 0 & \gamma & -3i\gamma^2 \\ 0 & i & -\gamma. \end{pmatrix}, \quad \mathrm{P} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1. \end{pmatrix}
$$

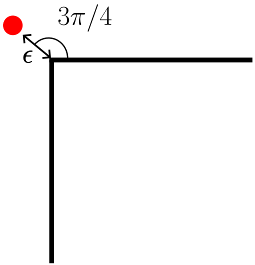
The problem is reduced to a 2×2 matrix problem and a scalar additive factorization problem.

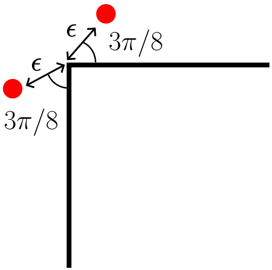
The submatrix

$$
\tilde{\mathrm{K}} = \frac{1}{2\gamma}\begin{pmatrix} \gamma & -3i\gamma^2 \\ i & -\gamma \end{pmatrix}
$$

is of Daniele-Khrapkov type.

Edge Green's functions





$$\mathrm{V}^{-}_{(1)}(z) = \mathrm{K}(z)\mathrm{V}^{+}_{(1)}(z) \qquad \mathrm{V}^{-}_{(2)}(z) = \mathrm{K}(z)\mathrm{V}^{+}_{(2)}(z)$$

$$\mathrm{X}^{\pm} = (\mathrm{V}^{\pm}_{(1)}, \mathrm{V}^{\pm}_{(2)})$$

Craster and Shanin, "Embedding formulae for diffraction by rational wedge and angular geometries"

Embedding formula

$$\tilde{\mathrm{U}}^{-}(z,z_i) = \frac{4z_i k \sin\theta^{\mathrm{in}}}{z^2 - z_i^2}\mathrm{X}^{-}(z)(\mathrm{X}^{-}(-z_i))^{-1}\begin{pmatrix}1\\0\end{pmatrix} + \frac{2k\sin\theta^{\mathrm{in}}}{z+z_i}\begin{pmatrix}1\\0\end{pmatrix},$$

Using the spectral relations

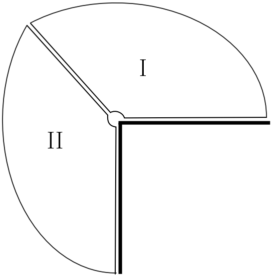
$$S(\theta,\theta^{\mathrm{in}}) = -k\sin\theta\, Y_1^{-}(-k\cos\theta), \quad S_j(\theta) = -k\sin\theta\, V_{2j}^{-}(-k\cos\theta).$$

$$S(\theta,\theta^{\mathrm{in}}) = \frac{4}{9\pi i k^2}\frac{S_1(\theta)S_2(\theta^{\mathrm{in}}) - S_1(\theta^{\mathrm{in}})S_2(\theta)}{\cos^2\theta - \cos^2\theta^{\mathrm{in}}}$$

Scalar Wiener–Hopf equation for the wedge

Apply Green's theorem twice to the domains *I* and *II* with

$$w = \exp\{ikr\cos(\theta - \psi(z))\}$$



$$S(\psi) + \Phi(\cos\psi) = \frac{-\sin\psi(z)}{\cos\psi - \cos\theta^{\mathrm{in}}},$$

$$S(\psi) + \Psi(\cos(\psi + \varphi)) = \frac{\sin(\varphi + \psi)}{\cos(\varphi + \psi) - \cos(\varphi + \theta^{\mathrm{in}})}, \quad \varphi = \frac{3\pi}{2}$$

$$S(\psi) = \int_\Gamma \left[\frac{\partial u^{\mathrm{sc}}}{\partial n} w - \frac{\partial w^{\mathrm{sc}}}{\partial n} u\right] dl,$$

$$\Phi(q) = \int_0^\infty \frac{\partial u^{\mathrm{sc}}}{\partial n}(r, 0) e^{ikrq} dr, \quad \Psi(q) = \int_0^\infty \frac{\partial u^{\mathrm{sc}}}{\partial n}(r, \varphi) e^{ikrq} dr,$$

Shanin, "Excitation of waves in a wedge-shaped region"

The change of variables

$$\Phi(\cos\psi(\alpha)) - \frac{\sin\theta^{\mathrm{in}}}{\cos\psi(\alpha) - \cos\theta^{\mathrm{in}}} = \Psi(\cos(\psi(\alpha)+\varphi)) - \frac{\sin(\theta^{\mathrm{in}}+\varphi)}{\cos(\varphi+\psi(\alpha)) - \cos(\varphi+\theta^{\mathrm{in}})}$$

$$\psi = \frac{3}{\pi}\arccos(\sqrt{\alpha})$$

Solution by pole removal

$$\Phi(\cos\psi(\alpha)) = \frac{r(\theta^{\mathrm{in}})}{\alpha - \alpha^{\mathrm{in}}}$$

Where does the embedding formula hide?

The solution by pole removal corresponds to the non-local embedding:

$$z(\alpha) = k\cos\psi(\alpha) = k(\alpha^{3/2} - 3\sqrt{\alpha})$$

Shanin and Craster, "Pseudo-differential operators for embedding formulae"

Getting the local embedding formula

Multiply the solution by a polynomial both in α and z that "kills" the pole α^{in}

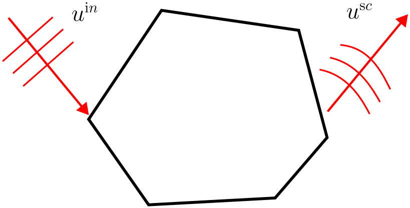
$$
\begin{aligned}
Q(z(\alpha), z_i) &= 4(2\alpha - 1)^3 - 3(2\alpha - 1) - \cos(2\theta^{\mathrm{in}}) \\
Q(z, z_i) &= 2(z/k)^2 - 1 - \cos(2\theta^{\mathrm{in}})
\end{aligned}
$$

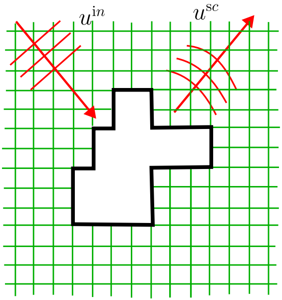
$$
\Phi(\cos\psi(\alpha)) = \frac{c_0(\theta^{\mathrm{in}}) + c_1(\theta^{\mathrm{in}})\alpha + c_2(\theta^{\mathrm{in}})\alpha^2}{Q(z(\alpha), z_i)}
$$

α, α^2 can be regarded as the edge Green's functions.

$$
S(\theta, \theta^{\mathrm{in}}) = -2i[\Phi(\cos(\theta - \pi)) - \Phi(\cos(\theta + \pi))]
$$

Embedding on lattices





$$\Delta u + k^2 u = 0$$
$$\Delta u = \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2}$$

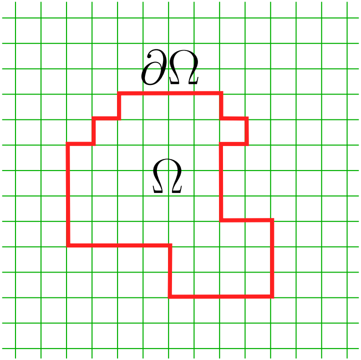
$$\Delta[u_{m,n}] + K^2 u_{m,n} = 0, \quad K = kh$$
$$\Delta[u_{m,n}] = u_{m+1,n} + u_{m-1,n}$$
$$+u_{m,n+1} + u_{m,n-1} - 4u_{m,n}$$

Can we transfer the results to the lattice diffraction problems?

Green's formula on lattices

$$\sum_{\nu\in\partial\Omega}(\delta[u_\nu]w_\nu-\delta[w_\nu]u_\nu)=\sum_{\nu\in\Omega}(f_\nu w_\nu-g_\nu u_\nu)\,,\qquad \nu=\{m,n\}$$

$$\begin{array}{rcl}\Delta[u]_\nu+K^2u_\nu & = & f_\nu\\ \Delta[w]_\nu+K^2w_\nu & = & g_\nu\end{array}$$



The analogy

Using the Green's theorem formalism Wiener-Hopf equations can be derived for problem on lattices, and the analogy can be established

	lattice	continuous
derivative	$\delta_{m,n}$	$\frac{\partial}{\partial n}$
kernel	$\Upsilon(q)$	$i\gamma(z)$
WH contour	unit circle	real axis
plane wave	$q^m l^n$	$\exp\{izx + i\gamma(z)y\}$
transforms	DFT	IFT
topology	torus	sphere

q, l are points on the dispersion surface of the lattice:

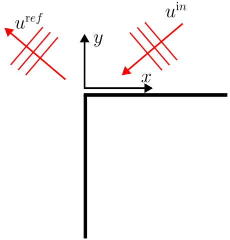
$$q + q^{-1} + l + l^{-1} + K^2 - 4 = 0$$
$$\Upsilon(q) = \sqrt{(q - \eta_{1,1})(q - \eta_{1,2})(q - \eta_{2,1})(q - \eta_{2,2})}$$

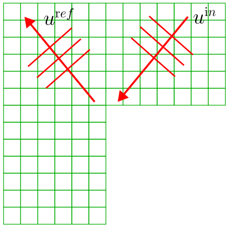
$\eta_{i,j}$ are some constants that depend on K

Example: Wedge problem on a lattice

Wiener–Hopf equation

$$\mathrm{U}^{-}(z, z_i) = \mathrm{K}(z)\mathrm{U}^{+}(z, z_i) + \mathrm{F}(z, z_i)$$



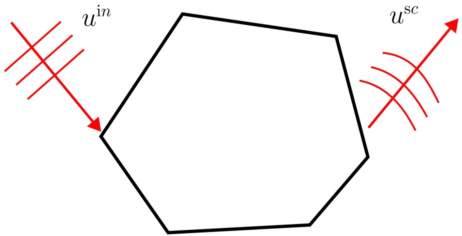


$$\tilde{\mathrm{K}}_c = \frac{1}{2\gamma}\begin{pmatrix} \gamma & -3i\gamma^2 \\ i & -\gamma \end{pmatrix} \qquad \tilde{\mathrm{K}}_l = \frac{1}{2\Upsilon}\begin{pmatrix} \Upsilon & -3\Upsilon^2 \\ -1 & -\Upsilon \end{pmatrix}$$

Then, the embedding formula is derived from the Wiener–Hopf perspective

Future work

Embedding in 2D for an arbitrary polygon[3] from the Wiener–Hopf perspective



Can we find an embedding formula that will work for irrational angles?

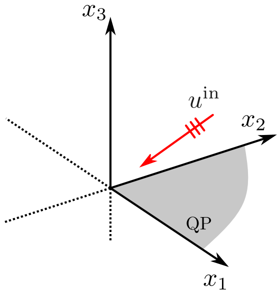
[3] Craster and Shanin, "Embedding formulae for diffraction by rational wedge and angular geometries".

What's about embedding in 3D? Quarter-plane problem

2D Wiener–Hopf equation

$$K(z_1, z_2)W(z_1, z_2) = U(z_1, z_2)$$

$$K(z_1, z_2) = \frac{1}{\sqrt{k^2 - z_1^2 - z_2^2}}$$



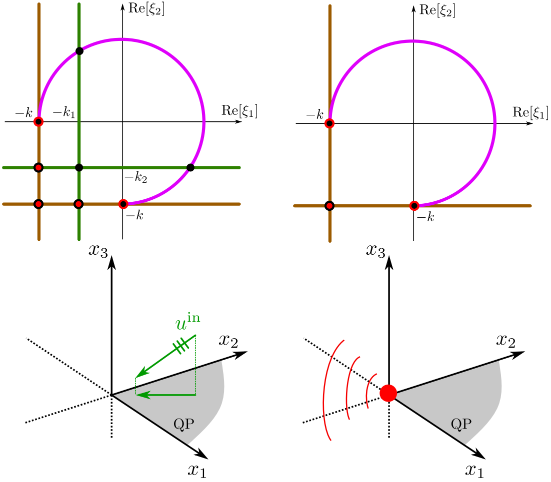
Analytic continuation

We don't now how to solve the equation (even formally) in 2D but we can use it as an analytic continuation formula

Assier and Shanin, "Diffraction by a quarter–plane. Analytical continuation of spectral functions"

Structure of the space of solutions

Singularities of the ;spectral function $U(z_1, z_2)$ for plane wave and point source problems

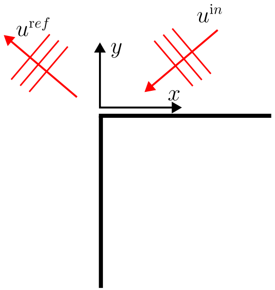


Is it possible to use the vertex Green's function as a basis function for the solution?

Example: the wedge problem

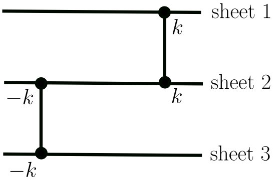
Using the Wiener–Hopf equation as analytic continuation formula one can find the Riemann surfaces of spectral functions (Wiener–Hopf unknowns):

$$Y_1^- = \int_{-\infty}^{0} \exp\{izx\} u_1^{\mathrm{sc}}(0, x) dx$$



The unknown is a function meromorphic on the given Riemann surface. Such functions form an algebraic field. The basis of the algebraic field is of dimension two:

$$[1, \cos(1/3 \arccos(-z/k))]$$



Then, the solution is expressed in terms of basis functions, which can be regarded as embedding.

Conclusions

There is a matrix Wiener–Hopf problem behind any embedding formula!