The Wiener–Hopf perspective on the embedding formula: new results

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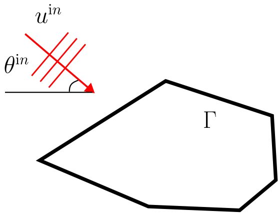
Outline

1. Standard approach to embedding
2. Novel approach to embedding. Wiener–Hopf perspective
3. Green's formula as Wiener–Hopf tool
4. The wedge problem revisited
5. Matrix Wiener–Hopf equation for the wedge
6. Embedding on lattices. New results!
7. Future work

What is embedding in wave theory?

$$(\Delta + k^2)u(x, y) = 0$$
$$u = u^{\mathrm{in}} + u^{\mathrm{sc}}$$
$$u|_\Gamma = 0$$



Study a differential operator $H[u]$ such that

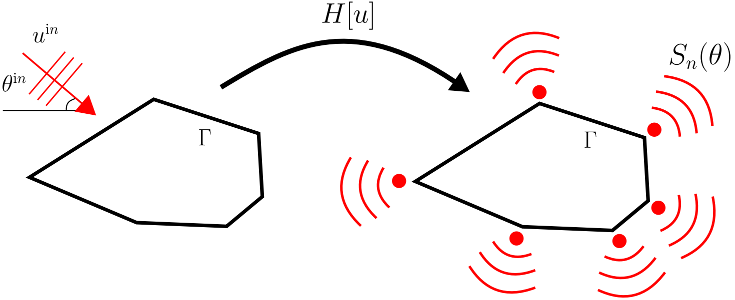
- $(\Delta + k^2)H[u] = H\left[(\Delta + k^2)u\right] = 0$, commutes with Helmholtz operator
- $H[u]$ preserves boundary conditions
- $H[u^{\mathrm{in}}] = 0$, kills the incident wave
- $H[u]$ doesn't satisfy vertex condition

By uniqueness, the following relation holds (**embedding formula**):

$$H[u] = \sum_{n=1}^{N} c_n(\theta^{\mathrm{in}})v_n(x, y),$$

v_n edge Green's functions, $c_n(\theta^{\mathrm{in}})$ some known coefficients.

Embedding for directivities



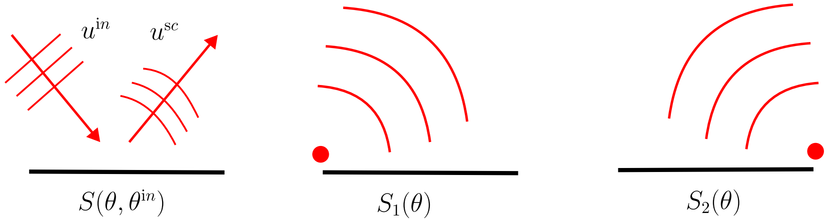
$u^{\text{sc}}(r,\theta) \approx -\frac{\exp\{ikr - i\pi/4\}}{\sqrt{2\pi kr}} S(\theta,\theta^{\text{in}}),\quad H[u] = q(k\cos\theta, k\cos\theta^{\text{in}})S(\theta^{\text{in}}),\quad q$ is a polynomial

Then, **embedding for directivities**[1] is

$$S(\theta,\theta^{\text{in}}) = \frac{\sum_{n=1}^{N} c_n(\theta^{\text{in}})S_n(\theta)}{q(k\cos\theta, k\cos\theta^{\text{in}})}$$

[1] Craster, Shanin, and Doubravsky, "Embedding formulae in diffraction theory".

Example: strip problem

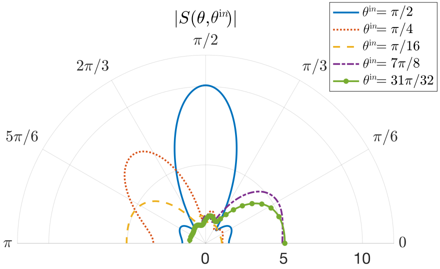


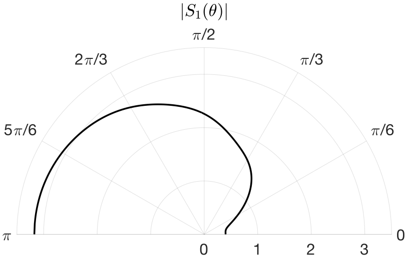
$$H = \frac{\partial}{\partial x} + ik\cos\theta^{\rm in}$$

$$S(\theta, \theta^{\rm in}) = \frac{S_2(\theta^{\rm in})S_2(\theta) - S_1(\theta^{\rm in})S_1(\theta)}{k\cos\theta + k\cos\theta^{\rm in}}$$

Computational benefit: Unknowns depend only on one variable

Strip: numerics





Knowing just $S_1(\theta)$ and $S_2(\theta) = S_1(\pi - \theta)$ we can $S(\theta, \theta^{\text{in}})$ for any θ^{in}.

Novel approach to embedding. Wiener–Hopf perspective

$$\mathrm{U}^{-}(t, z^{\mathrm{in}}) = \mathrm{K}(t)\mathrm{U}^{+}(t, z^{\mathrm{in}}) + \frac{\mathrm{r}}{z - z^{\mathrm{in}}}, \quad t \in \mathbb{R} \quad (1)$$

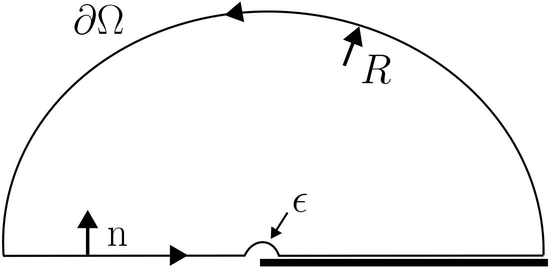
For polar forcing embedding follows from the method of normal solutions[2]:

- Find N linearly independent solutions of (1) without a forcing term
- Compose a matrix $\mathrm{X}^{\pm}$ of solutions. $\mathrm{X}^{\pm}$ is called the matrix of normal solutions if $\det|\mathrm{X}^{\pm}| = 0$.
- Factorise the kernel $\mathrm{K}(z) = \mathrm{X}^{-}(z)(\mathrm{X}^{+}(z))^{-1}$
- Apply Liouville's theorem to get the solution:

$$\mathrm{U}^{+}(z, z^{\mathrm{in}}) = -\frac{\mathrm{X}^{+}(z)(\mathrm{X}^{-}(z^{\mathrm{in}}))^{-1}\mathrm{r}}{z - z^{\mathrm{in}}}$$

[2] Gakhov, "Riemann's boundary problem for a system of n pairs of functions".

Green's formula as Wiener–Hopf tool



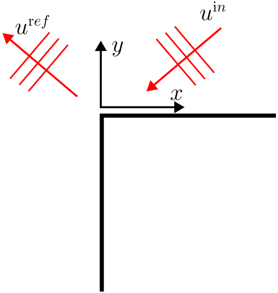
$$\int_{\partial\Omega} \left[w \frac{\partial u^{\mathrm{sc}}}{\partial n} - u^{\mathrm{sc}} \frac{\partial w}{\partial n} \right] dl = 0$$

$$w = \exp\left\{ izx + i\gamma(z) y \right\}, \quad \gamma(z) = \sqrt{k^2 - z^2}$$

The integral on Ω tends to zero due to radiation conditions, and the rest gives us a functional relation between "plus" and "minus" spectral functions

The wedge problem

$$(\Delta + k^2)u(x,y) = 0$$
$$u = u^{\mathrm{in}} + u^{\mathrm{ref}} + u^{\mathrm{sc}}$$
$$u\big|_{\mathrm{wedge}} = 0$$



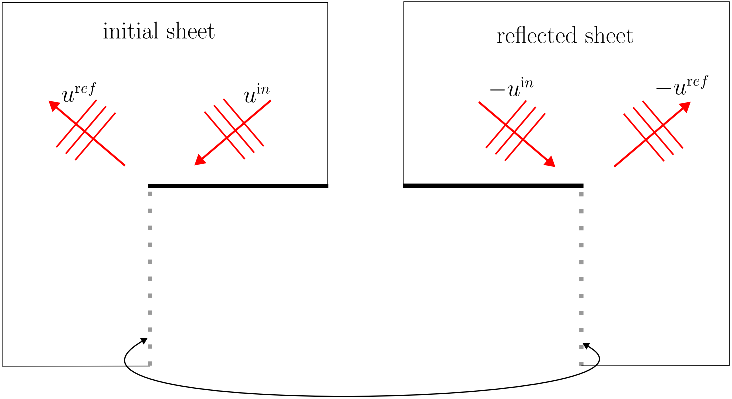
Near-field behaviour:

$$u^{\mathrm{sc}} = Cr^{2/3}\sin(2/3\theta) + O(r^{4/3})$$

Far-field behaviour:

$$u^{\mathrm{sc}}(r,\theta) = -\frac{\exp\{ikr - i\pi/4\}}{\sqrt{2\pi kr}} S(\theta, \theta^{\mathrm{in}}) + O\left(\frac{1}{r}\right).$$

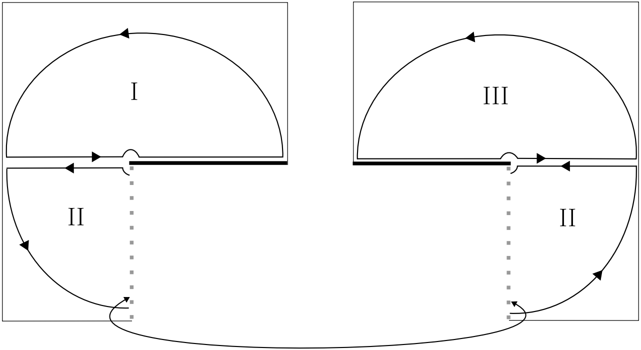
Reformulation on a branched manifold with a boundary



Apply Green's theorem 3 times with

$$w_1 = \exp\{izx + i\gamma(z)y\}$$
$$w_2 = \exp\{izx - i\gamma(z)y\}$$

Reformulation on a branched manifold with a boundary



$$Y_i^- = \int_{-\infty}^{0} \exp\{izx\} u_i^{\mathrm{sc}}(0,x)dx, \quad W_i^- = \int_{-\infty}^{0} \exp\{izx\} \frac{\partial u_i^{\mathrm{sc}}}{\partial y}(-0,x)dx,$$

$$Y_i^+ = \int_{0}^{\infty} \exp\{izx\} u_i^{\mathrm{sc}}(0,x)dx, \quad W_i^+ = \int_{0}^{\infty} \exp\{izx\} \frac{\partial u_i^{\mathrm{sc}}}{\partial y}(-0,x)dx.$$

Wiener-Hopf equation

$$\mathrm{U}^{-}(z, z^{\mathrm{in}}) = \mathrm{K}(z)\mathrm{U}^{+}(z, z^{\mathrm{in}}) + \mathrm{F}(z, z^{\mathrm{in}})$$

$$\mathrm{U}^{-}(z, z^{\mathrm{in}}) = \begin{pmatrix} W_2^{-}(z, z^{\mathrm{in}}) \\ W_1^{-}(z, z^{\mathrm{in}}) \\ Y_1^{-}(z, z^{\mathrm{in}}) \end{pmatrix}, \quad \mathrm{U}^{+}(z, z^{\mathrm{in}}) = \begin{pmatrix} W_1^{+}(z, z^{\mathrm{in}}) \\ W_2^{+}(z, z^{\mathrm{in}}) \\ Y_2^{+}(z, z^{\mathrm{in}}) \end{pmatrix}$$

$$\mathrm{K} = \frac{i}{2\gamma} \begin{pmatrix} 0 & 2i\gamma & 2\gamma^2 \\ i\gamma & i\gamma & -\gamma^2 \\ -1 & 1 & i\gamma \end{pmatrix}, \quad \mathrm{F} = -\begin{pmatrix} \dfrac{2k \sin\theta^{\mathrm{in}}}{z + z^{\mathrm{in}}} \\ \dfrac{k \sin\theta^{\mathrm{in}}}{(z - z^{\mathrm{in}})} \\ \dfrac{ik \sin\theta^{\mathrm{in}}}{(z - z^{\mathrm{in}})\gamma} \end{pmatrix}.$$

Aitken, "On the factorisation of matrix Wiener–Hopf kernels arising from acoustic scattering problems"

Reducing the equation

$$\mathrm{K} \to \mathrm{PKP}^{-1} = \frac{1}{2\gamma}\begin{pmatrix} -1 & 0 & 0 \\ 0 & \gamma & -3i\gamma^2 \\ 0 & i & -\gamma \end{pmatrix}, \quad \mathrm{P} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

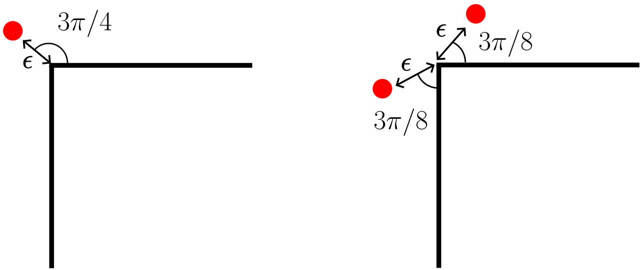
The problem is reduced to a 2×2 matrix problem and a scalar additive factorization problem.

The submatrix

$$\tilde{\mathrm{K}} = \frac{1}{2\gamma}\begin{pmatrix} \gamma & -3i\gamma^2 \\ i & -\gamma \end{pmatrix}$$

is of Daniele-Khrapkov type.

Edge Green's functions



$$\mathrm{V}^-_{(1)}(z) = \mathrm{K}(z)\mathrm{V}^+_{(1)}(z) \qquad \mathrm{V}^-_{(2)}(z) = \mathrm{K}(z)\mathrm{V}^+_{(2)}(z)$$

$$\mathrm{X}^\pm = (\mathrm{V}^\pm_{(1)}, \mathrm{V}^\pm_{(2)})$$

Craster and Shanin, "Embedding formulae for diffraction by rational wedge and angular geometries"

Embedding formula

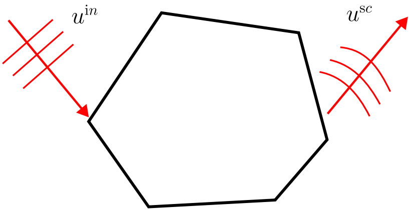
$$\tilde{\mathrm{U}}^{-}(z, z^{\mathrm{in}}) = \frac{4z^{\mathrm{in}} k \sin\theta^{\mathrm{in}}}{z^2 - (z^{\mathrm{in}})^2} \mathrm{X}^{-}(z)(\mathrm{X}^{-}(-z^{\mathrm{in}}))^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2k \sin\theta^{\mathrm{in}}}{z + z^{\mathrm{in}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

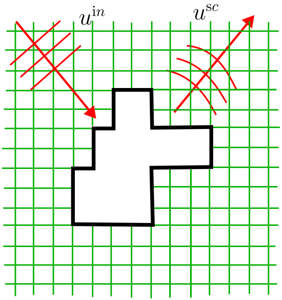
Using the spectral relations

$$S(\theta, \theta^{\mathrm{in}}) = -k \sin\theta \, Y_1^{-}(-k\cos\theta), \quad S_j(\theta) = -k\sin\theta \, V_{2j}^{-}(-k\cos\theta).$$

$$S(\theta, \theta^{\mathrm{in}}) = \frac{4}{9\pi i k^2} \frac{S_1(\theta) S_2(\theta^{\mathrm{in}}) - S_1(\theta^{\mathrm{in}}) S_2(\theta)}{\cos^2\theta - \cos^2\theta^{\mathrm{in}}}$$

Embedding on lattices: new results





$$\Delta u + k^2 u = 0$$
$$\Delta u = \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2}$$

$$\Delta_{m,n}[u] + K^2 u_{m,n} = 0, \quad K = kh$$
$$\Delta[u_{m,n}] = u_{m+1,n} + u_{m-1,n}$$
$$+u_{m,n+1} + u_{m,n-1} - 4u_{m,n}$$

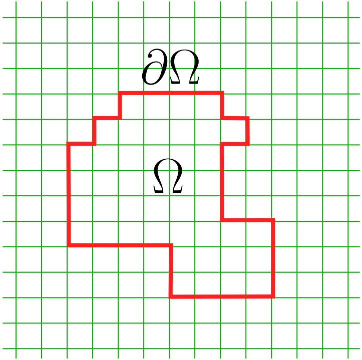
Can we transfer the results to the lattice diffraction problems?

On an analogy between continuous and lattice problems

Green's formula on lattices

$$\sum_{\nu\in\partial\Omega}\left(\delta_\nu[u]w_\nu - \delta_\nu[w]u_\nu\right) = \sum_{\nu\in\Omega}\left(f_\nu w_\nu - g_\nu u_\nu\right), \qquad \nu = \{m, n\}$$

$$\begin{aligned} \Delta_\nu[u] + K^2 u_\nu &= f_\nu \\ \Delta_\nu[w] + K^2 w_\nu &= g_\nu \end{aligned}$$



The analogy

Using Green's theorem formalism Wiener-Hopf equations can be derived for problems on lattices, and the analogy can be established

	lattice	continuous
derivative	$\delta_{m,n}$	$\frac{\partial}{\partial n}$
kernel	$\Upsilon(s)$	$i\gamma(z)$
WH contour	unit circle	real axis
plane wave	$s^m q^n$	$\exp\{izx + i\gamma(z)y\}$
transforms	DFT	IFT
topology	torus	sphere

Lattice dispersion equation:

$$s + s^{-1} + q + q^{-1} + K^2 - 4 = 0$$

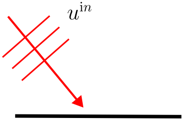
$$\Upsilon(s) = \sqrt{(s - \eta_{1,1})(s - \eta_{1,2})(s - \eta_{2,1})(s - \eta_{2,2})}$$

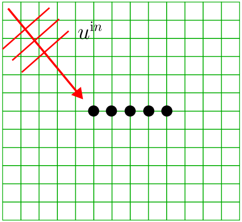
$\eta_{i,j}$ are some constants that depend on K.
Continuous dispersion equation:

$$z^2 + \gamma(z)^2 = k^2$$

Example: strip problem

$$\mathrm{U}^-(z, z^{\mathrm{in}}) = \mathrm{K}(z)\mathrm{U}^+(z, z^{\mathrm{in}}) + \mathrm{F}(z, z^{\mathrm{in}})$$





$$\mathrm{K}_c(z) = \begin{pmatrix} -\exp\{2iza\} & (i\gamma(z))^{-1} \\ 0 & \exp\{-2iza\} \end{pmatrix}, \quad \mathrm{K}_d(s) = \begin{pmatrix} -s^{-2N} & \Upsilon(s)^{-1} \\ 0 & s^{2N} \end{pmatrix}$$

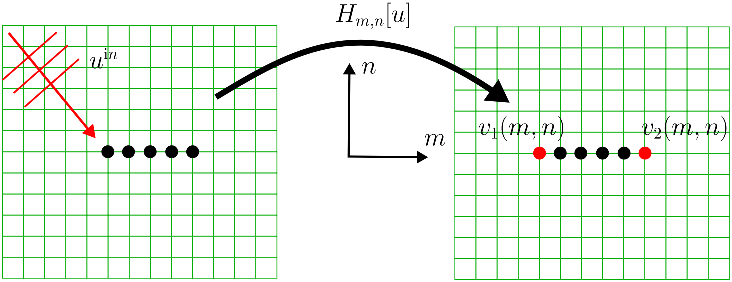
Formally, we get the embedding from Wiener–Hopf perspective (**new result**):

$$\mathrm{U}_c^+(z, z^{\mathrm{in}}) = -\frac{\mathrm{X}^+(z)(\mathrm{X}^-(z^{\mathrm{in}}))^{-1}\mathrm{r}_1}{z - z^{\mathrm{in}}}, \quad \mathrm{U}_d^+(s, s^{\mathrm{in}}) = -\frac{\mathrm{X}^+(s)(\mathrm{X}^-(s^{\mathrm{in}}))^{-1}\mathrm{r}_2}{s - s^{\mathrm{in}}}$$

Operator approach for the strip on a lattice

Consider the operator

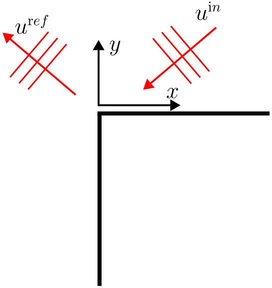
$$H_{m,n}[u] = u(m+1,n) - s^{\mathrm{in}} u(m,n), \quad s^{\mathrm{in}} = \exp\{iK\cos\theta^{\mathrm{in}}\}$$

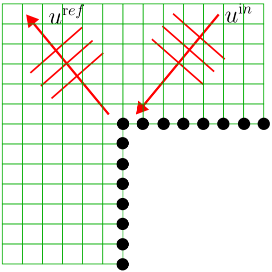


Embedding for the fields:

$$H_{m,n}[u] = c_1(s^{\mathrm{in}})v_1(m,n) + c_2(s^{\mathrm{in}})v_2(m,n)$$

Example: Wedge problem on a lattice





$$\tilde{\mathrm{K}}_c = \frac{1}{2\gamma}\begin{pmatrix} \gamma & -3i\gamma^2 \\ i & -\gamma \end{pmatrix}$$

$$\tilde{\mathrm{K}}_l = \frac{1}{2\Upsilon}\begin{pmatrix} \Upsilon & -3\Upsilon^2 \\ -1 & -\Upsilon \end{pmatrix}$$

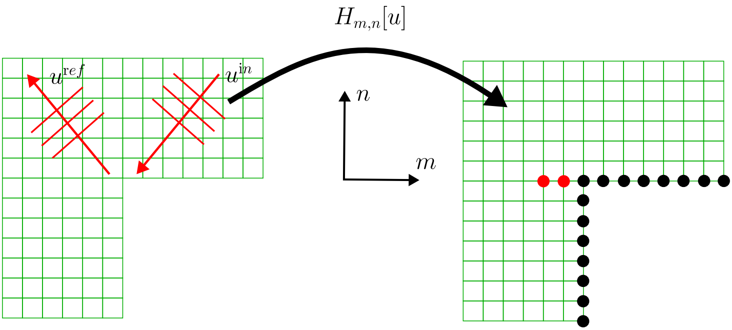
Embedding for the wedge on a lattice (**new result**):

$$\tilde{\mathrm{U}}_d^- = \frac{2s\left(s^{\mathrm{in}} - (s^{\mathrm{in}})^{-1}\right)\Upsilon(s^{\mathrm{in}})}{s^{\mathrm{in}}(s^2 - s(s^{\mathrm{in}} - (s^{\mathrm{in}})^{-1}) + 1)}\mathrm{X}^-(s)\left(X^-((s^{\mathrm{in}})^{-1})\right)^{-1}\begin{pmatrix}1\\0\end{pmatrix} + \frac{2s\Upsilon(s^{\mathrm{in}})}{s^{\mathrm{in}}(s - (s^{\mathrm{in}})^{-1})}\begin{pmatrix}1\\0\end{pmatrix}$$

Operator approach for the wedge on a lattice

Consider the operator

$$H_{m,n}[u] = u(m+2,n) + u(m,n) - (s^{\mathrm{in}} - (s^{\mathrm{in}})^{-1})u(m+1,n)$$

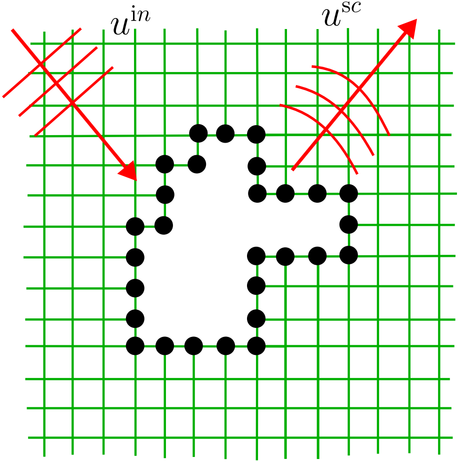


Embedding for the fields:

$$H_{m,n}[u] = c_1(s^{\mathrm{in}})v_1(m,n) + c_2(s^{\mathrm{in}})v_2(m,n)$$

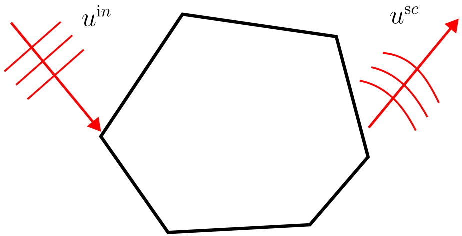
Bonus: Embedding for an arbitrary shape on a lattice

Knowing the embedding formula for the right angle and the strip we can find a general embedding formula for an arbitrary shape.



Future work

Embedding in 2D for an arbitrary polygon[3] from the Wiener–Hopf perspective



Can we find an embedding formula that will work for irrational angles?

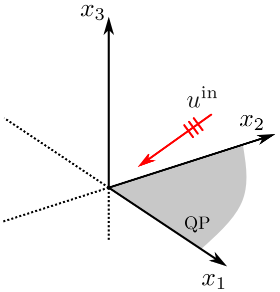
[3] Craster and Shanin, "Embedding formulae for diffraction by rational wedge and angular geometries".

What's about embedding in 3D? Quarter-plane problem

2D Wiener–Hopf equation

$$U(z_1, z_2) = K(z_1, z_2)W(z_1, z_2) + \frac{1}{(z_1 - z_1^{\mathrm{in}})(z_2 - z_2^{\mathrm{in}})}$$

$$K(z_1, z_2) = \frac{1}{\sqrt{k^2 - z_1^2 - z_2^2}}$$



Analytic continuation

We don't know how to solve the equation (even formally) in 2D but we can use it as an analytic continuation formula

Assier and Shanin, "Diffraction by a quarter–plane. Analytical continuation of spectral functions"

Conclusions

There is a matrix Wiener–Hopf problem behind any embedding formula!

Preprint is available on arXiv https://arxiv.org/pdf/2410.08684